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ANALYTIC GEOMETRY

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PREFACE

The first edition of this book was designed primarily to provide a short course in Analytic Geometry, which would include all the topics with which students of the ordinary text in Calculus are supposed to be familiar and which could at the same time be shortened without difficulty. This purpose largely controlled both the selection and the arrangement of the contents.

Since its publication the book has been used in a number of colleges, much to the gratification of the authors, and as a result of this use various suggestions as to improvements have been received. It has seemed desirable, therefore, to attempt a revision of the book, which, it is hoped, will remedy the defects without sacrificing the merits of the old edition.

The most conspicuous changes are the addition of a chapter on graphs of functions and empirical equations and a large increase in the number of problems, both of which have been requested by several teachers. Answers to the odd numbered problems have been included in the book, so that problems can be assigned with or without answers, as may be desired, and so that the student can work by himself to advantage. Minor changes are chiefly in the nature of simplification of the text and additional explanatory material.

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CONTENTS

SECTION PAGE

INTRODUCTION xiii

Definitions and Formulas from Algebra and Trigonometry

CHAPTER I — CARTESIAN COÖRDINATES

1. Introduction 1
2. Position; Coördinates 1
3. Cartesian Coördinates 2
4. Notation 2
5. Plotting of Points 3
6. Directed Lines 6
7. Projections of a Directed Segment on the Coördinate Axes 7
8. Distance Formula 8
9. Point of Division Formulas 9
10. Applications to Geometric Proofs 11
11. Angle between Two Lines 14
12. Inclination and Slope 15
13. The Slope Formula 15
14. Applications of the Slope Formula 16
15. Parallelism and Perpendicularity 17
16. The Angle Formula 18

CHAPTER II — CURVES AND EQUATIONS

17. Introduction 22
18. Plotting of Graphs 22
19. Equations Containing Arbitrary Constants 26
20. Derivation of Equations 27
21. Functional Variables 33
22. Functional Notation 34
23. Discussion of Curves 36
24. Intercepts 36

SECTION PAGE
25. Symmetry 36
26. Extent 39
27. Horizontal and Vertical Asymptotes 42
28. Symmetrical Transformations 45
29. Transformations of Equations not Altering Loci 46
30. Equations Whose Graphs cannot be Plotted 47
31. Intersection of Curves 48

CHAPTER III — THE STRAIGHT LINE

32. Equations of the Straight Line 51
33. The Point Slope and Two-Point Forms 51
34. The Slope Intercept Form 52
35. The Intercept Form 52
36. Lines Parallel to the Axes 53
37. The Linear Equation 56
38. Relations between Linear Equations 57
39. Application to the Solution of Linear Equations 60
40. The Normal Form 64
41. Reduction of any Linear Equation to the Normal Form 65
42. Distance from a Line to a Point 68
43. Geometric Conditions Determining a Line 73
44. Systems of Straight Lines 74
45. Systems of Lines through the Intersection of Two Given Lines 76
46. Plotting by Factoring 79

CHAPTER IV — THE CIRCLE

47. The Standard Equation of the Circle 82
48. The General Equation of the Circle 85
49. Identification of Center and Radius 85
50. Special Forms 86
51. The Equation of the Circle Derived from Three Conditions 89
52. Length of a Tangent 93
53. Common Chord of Two Circles 94
54. Radical Axis 96
55. Systems of Circles 97

CHAPTER V — THE PARABOLA

SECTION PAGE
56. Definition 101
57. The Equation of the Parabola 101
58. Discussion of the Parabola 103
59. Latus Rectum 103
60. Construction of the Parabola 103
61. Properties and Applications 105

CHAPTER VI — THE ELLIPSE

62. Definition 111
63. The Equation of the Ellipse 111
64. Discussion of the Ellipse 112
65. Focal Radii 113
66. Latus Rectum 113
67. Ellipse with Foci on Y-axis 116
68. Limiting Forms of the Ellipse 117
69. Eccentricity 117
70. Construction of an Ellipse 117
71. Applications 118

CHAPTER VII — THE HYPERBOLA

72. Definition 122
73. The Equation of the Hyperbola 122
74. Discussion of the Hyperbola 123
75. Latus Rectum 124
76. Hyperbola with Foci on the Y-axis 124
77. Focal Radii 124
78. Eccentricity 127
79. Asymptotes 127
80. Conjugate Hyperbolas 128
81. Equilateral Hyperbolas 130
82. Construction of a Hyperbola 131
83. Applications 131

CHAPTER VIII — TRANSFORMATION OF COÖRDINATES AND SIMPLIFICATION OF EQUATIONS

84. Change of Axes 135
85. Formulas of Translation 135

SECTION PAGE
86. Formulas of Rotation 136
87. Test for Axes of Symmetry 139
88. Simplification of Equations by Translation 139
89. Discussion of the Equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ 143
90. Conics 144
91. General Statement 146
92. Generalized Standard Equations of the Conics 146
93. Simplification of Equations by Rotation 151
94. The General Equation of the Second Degree 152
95. The Characteristic 153
96. Test for Distinguishing the Conics 154
97. Suggestions for Simplifying the Equation of a Conic . 154
98. The Conic through Five Points 156

CHAPTER IX — POLAR COÖRDINATES

99. Preliminary Note 159
100. Definitions 159
101. Relations between Rectangular and Polar Coördinates 161
102. Polar Curves 162
103. Equations of the Straight Line 164
104. Equations of the Circle 165
105. Discussion of Polar Curves 166
106. Rotation of Axes 170
107. Symmetrical Transformations 171
108. Polar Equation of the Conics 172
109. Equations derived in Polar Coördinates 176
110. Review 178

CHAPTER X — HIGHER PLANE CURVES

111. Algebraic and Transcendental Equations 180
112. The Exponential Curve 180
113. The Logarithmic Curve 181
114. Applications 183
115. Periodic Functions 186
116. The Sine Curve 186
117. Circular Measure 187
118. The Cosine Curve 188
119. Multiple Angles 189

SECTION PAGE
120. Sum of Functions 190
121. The Tangent Curve 191
122. Inverse Trigonometric Functions 192
123. Parametric Equations 194
124. Parametric Equations of the Circle 195
125. Parametric Equations of the Ellipse 195
126. Path of a Projectile 196
127. The Cycloid 200
128. The Epicycloid and Hypocycloid 202
129. The Involute of the Circle 204
130. Algebraic Curves 206
131. The Witch 206
132. The Strophoid 207
133. The Ovals of Cassini 208

CHAPTER XI — TANGENTS AND NORMALS

134. Definitions 210
135. Slope of Tangent and Normal to a Circle at a Given Point 210
136. Slope of Tangent and Normal to an Ellipse at a Given Point 211
137. Slope of Tangent and Normal to the Parabola and Hyperbola at a Given Point 212
138. Equations of the Tangent and Normal at a Given Point 213
139. Tangent and Normal. Subtangent and Subnormal . 215
140. Tangent having a Given Slope 216
141. Tangent from a Given External Point 217
142. The Parabolic Reflector 219
143. Angle between the Focal Radii and the Tangent to an Ellipse at any Point 220
144. Diameters of a Conic 221
145. Conjugate Diameters of an Ellipse 222
146. Conjugate Diameters of the Hyperbola 223

CHAPTER XII — GRAPHS OF FUNCTIONS AND EMPIRICAL EQUATIONS

147. Functions in General 226
148. General Problems 227

SECTION PAGE
149. Plotting from Tabulated Data 227
150. Plotting from Formulas 229
151. The Graph as an Aid in Computing 230
152. Derivation of Equations 234
153. A Geometrical Problem 235
154. Linear Functions 235
155. Variation, Direct and Inverse 237
156. Several Common Algebraic Functions 238
157. Empirical Formulas 242
158. General Procedure 243
159. The Type $y = b + mx$ 244
160. Principle Involved in the Method of Averages 246
161. First Differences 247
162. Equations of the forms $y = a + bx^2$ and $y = a + b/x$ 249
163. The Parabola $y = a + bx + cx^2$ 251
164. Equations of the form $y = kx^n$ 254
165. Equations of the form $y = ke^{nx}$ 255
166. Conclusion 257

CHAPTER XIII — SOLID ANALYTIC GEOMETRY

167. Introduction 259
168. Coördinates 259
169. Radius Vector and Direction Cosines 260
170. Distance between Two Points 261
171. Direction of a Line 262
172. Angle between Two Lines 263
173. The Locus in Solid Geometry 265
174. The Normal Equation of the Plane 266
175. Plane Parallel to One or More Coördinate Axes 267
176. The General Equation of the First Degree 267
177. Angle between Two Planes 268
178. The Intercept Equation 269
179. The Equations of the Line 271
180. The Symmetrical Equations 272
181. The Two-point Equations 272
182. The Projection Forms 273
183. Reduction of the General Equations of a Line to the Standard Forms 273

SECTION PAGE
184. Cylindrical Surfaces 275
185 Surfaces of Revolution 277
186. Discussion of Surfaces 279
187. Quadric Surfaces 281
188. The Ellipsoid 281
189. Hyperboloid of One Sheet 282
190. Hyperboloid of Two Sheets 283
191. Elliptic Paraboloid 284
192. Hyperbolic Paraboloid 285

APPENDIX:
Plane Sections of a Right Circular Cone 288
Formulas and Equations 292

INTRODUCTION

DEFINITIONS AND FORMULAS FROM ALGEBRA AND TRIGONOMETRY

1. The Quadratic. — An expression reducible to the form

$$Ax^2 + Bx + C,$$

where A, B, and C are any constants, is called a quadratic in x.

2. Solution of the Quadratic. — The roots of any quadratic equation may be found by completing the square, or by using the quadratic formula.

(*a*) To solve by completing the square, transpose the constant term to the right-hand member and divide both sides of the equation by the coefficient of x^2. In the resulting equation add to both members the square of half the coefficient of x and extract the square root.

(*b*) If the above rule is applied to the general quadratic $Ax^2 + Bx + C = 0$, we get

$$x = -\frac{B \pm \sqrt{B^2 - 4\,AC}}{2\,A},$$

which is known as the *quadratic formula.* To solve an equation by means of this formula simply substitute for A, B, and C in the formula their values in the given equation.

3. The Discriminant.—The expression under the radical, $B^2 - 4\,AC$, is called the *discriminant* of the quadratic. It is denoted by the Greek letter Δ. The quadratic formula shows that the roots of a quadratic equation are

$$\begin{array}{ll} \text{imaginary,} & \text{if } \Delta < 0; \\ \text{real and equal,} & \text{if } \Delta = 0; \\ \text{real and unequal,} & \text{if } \Delta > 0. \end{array}$$

4. Logarithms. — The logarithm of a number to a given base is that exponent by which the base must be affected in order that the result may be equal to the number. Thus if $b^x = N$, we say x is the logarithm of N to the base b and write $x = \log_b N$.

The base b may be any positive number different from 0 and 1. Two systems of logarithms are in common use: the natural, or Naperian, where the base is $e = 2.71828 \ldots$; and the common, or Briggs, where the base is 10.

Important properties of logarithms to any base are the following:

$$\log_b 1 = 0; \qquad \log_b b = 1;$$

$$\log_b MN = \log_b M + \log_b N; \qquad \log_b \frac{M}{N} = \log_b M - \log_b N;$$

$$\log_b N^k = k \log_b N; \qquad \log_b \sqrt[k]{N} = \frac{1}{k} \log_b N;$$

$$\log_c N = \frac{\log_b N}{\log_b c} = \log_b N \cdot \log_c b.$$

5. Angles. — In trigonometry an angle is supposed to be generated by revolving a line, called the *generating line*, about the vertex from one side of the angle, called the *initial line*, to the other side, called the *terminal line*. If the rotation is counter-clockwise, the angle is called positive; if clockwise, it is called negative.

Degree Measure. — The ordinary unit of angle measurement is one ninetieth part of a right angle, which is called a *degree*. One sixtieth part of a degree is called a minute and one sixtieth part of a minute is called a second. Thus

$60'' =$ one minute;
$60' =$ one degree;
$360° =$ one complete revolution $=$ four right angles.

Circular Measure. — The unit of circular measure is a central angle subtended by an arc equal to the radius of the circle. It is called a *radian*. We find at once that

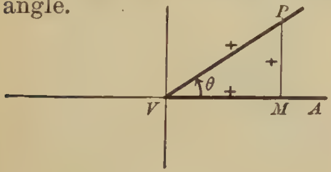
$$\pi \text{ radians} = 180°;$$

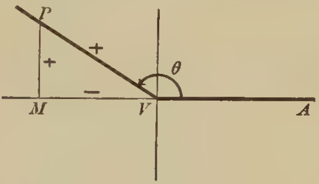
$$1 \text{ radian} = \frac{180°}{\pi};$$

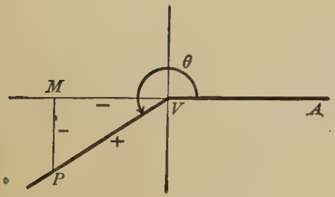
$$1° = \frac{\pi}{180} \text{ radians.}$$

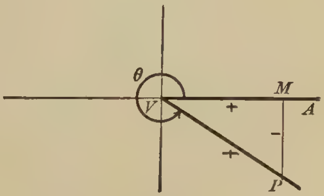
From this definition we have the important result that in any circle the length of an arc equals the product of the measure of its subtended central angle in radians and the length of the radius.

6. Trigonometric Functions. — From any point P of the terminal line VP of the angle $\theta = AVP$ drop a perpendicular on the initial line of the angle (produced if necessary). The ratios of the sides of the triangle thus formed are called the trigonometric ratios or functions of the given angle.









They are

$$\sin\theta = \frac{MP}{VP};\qquad \cos\theta = \frac{VM}{VP};\qquad \tan\theta = \frac{MP}{VM};$$

$$\csc\theta = \frac{VP}{MP};\qquad \sec\theta = \frac{VP}{VM};\qquad \cot\theta = \frac{VM}{MP}.$$

In these ratios the side of the triangle opposite the angle is considered positive when it extends up from the initial line, negative when down; the adjacent side is considered positive if it extends to the right of the vertex, negative if to the left; the hypothenuse is always positive.

7. Relations between the Functions. — For any angle,

$$\sec\theta = \frac{1}{\cos\theta};\qquad \cot\theta = \frac{\cos\theta}{\sin\theta};$$

$$\csc\theta = \frac{1}{\sin\theta};\qquad \sin^2\theta + \cos^2\theta = 1;$$

$$\cot\theta = \frac{1}{\tan\theta};\qquad 1 + \tan^2\theta = \sec^2\theta;$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta};\qquad 1 + \cot^2\theta = \csc^2\theta.$$

8. Reduction of Angles. —

$$\sin(-\theta) = -\sin\theta;\qquad \sin\left(\frac{\pi}{2} \pm \theta\right) = \cos\theta;$$

$$\cos(-\theta) = \cos\theta;\qquad \cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin\theta;$$

$$\tan(-\theta) = -\tan\theta;\qquad \tan\left(\frac{\pi}{2} \pm \theta\right) = \mp\cot\theta;$$

$$\sin(\pi \pm \theta) = \mp\sin\theta;$$

$$\cos(\pi \pm \theta) = -\cos\theta;$$

$$\tan(\pi \pm \theta) = \pm\tan\theta.$$

9. Special Angles. —

Angle	0	30 $\frac{\pi}{6}$	45 $\frac{\pi}{4}$	60 $\frac{\pi}{3}$	90 $\frac{\pi}{2}$	180 π	270 $\frac{3\pi}{2}$	36 2π
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0

10. Formulas for the Sum and Difference of Two Angles. —

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi;$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi;$$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}.$$

11. Double and Half Angle Formulas. —

$$\sin 2\theta = 2\sin\theta\cos\theta;$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1;$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta};$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}; \quad \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}};$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}.$$

12. Triangle Formulas. — In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{(Law of Sines)};$$

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{(Law of Cosines)}.$$

13. **The Greek Alphabet —**

LETTERS	NAMES	LETTERS	NAMES
Α, α	Alpha	Ν, ν	Nu
Β, β	Beta	Ξ, ξ	Xi
Γ, γ	Gamma	Ο, o	Omicron
Δ, δ	Delta	Π, π	Pi
Ε, ϵ	Epsilon	Ρ, ρ	Rho
Ζ, ζ	Zeta	Σ, σ	Sigma
Η, η	Eta	Τ, τ	Tau
Θ, θ	Theta	Υ, υ	Upsilon
Ι, ι	Iota	Φ, ϕ	Phi
Κ, κ	Kappa	Χ, χ	Chi
Λ, λ	Lambda	Ψ, ψ	Psi
Μ, μ	Mu	Ω, ω	Omega

ANALYTIC GEOMETRY

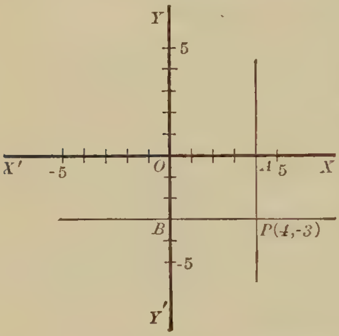
CHAPTER I

CARTESIAN COORDINATES

1. Introduction. — The chief feature of Analytic Geometry, which distinguishes it from the geometry hitherto familiar to the student, is its extensive use of algebraic methods in the solution of geometric problems. Just as the use of symbols in algebra makes possible the ready solution of many problems which would be difficult if not impossible by the processes of arithmetic, so the use of algebraic reasoning simplifies much of geometry and widens its scope. The student is already familiar with some of these applications; for example, the theorems stating the numerical properties of a triangle are derived in whole or part by algebraic reasoning.

2. Position; Coördinates. — A relation exists between the numbers of algebra and the points of geometry whereby to any point in the plane correspond two numbers, and conversely, any two numbers determine a point in the plane. Any system of representing the position of points in a plane by a pair of numbers is known as a *system of coördinates.* The two numbers may represent the distance and the direction of a point from a fixed reference point; or they may represent the respective distances of a point from two fixed perpendicular lines of reference. Both of these methods are commonly used in ordinary life. Thus we say, Baltimore is 40 miles northeast (N. 45° E.) of Washington; or we say — go 5 blocks east of A Street and 3 blocks south of First Avenue to reach the library. Each of these systems will be developed analytically so that algebraic equations will represent geometric loci, and conversely geometric loci will be represented by algebraic equations. The second method is simpler and will be discussed first.

3. Cartesian Coördinates. — In the figure the reference lines are the perpendiculars $X'X$ and $Y'Y$. These are called the **axes of coördinates** or **coördinate axes,** and their intersection O is called the **origin.** The two axes divide the plane into four quadrants numbered as in trigonometry. The position of a point P is determined by measuring its distance from $Y'Y$ along a parallel to $X'X$, and its distance from $X'X$ along a parallel to $Y'Y$. Distances measured to the right of $Y'Y$ or up from $X'X$ are called *positive,* those to the left or down *negative.* The measures of these distances, with the proper signs prefixed, are called the **coördinates** of the point; the one measured along a line parallel to $X'X$ is the **x-coördinate** or **abscissa**; the one parallel to $Y'Y$ is the **y-coördinate** or **ordinate.** Thus in the figure the abscissa of P is $BP = +4$, its ordinate is $AP = -3$. The coördinate axis $X'X$ is called the *axis of abscissas* or **x-axis,** and the axis $Y'Y$ the *axis of ordinates* or **y-axis.**

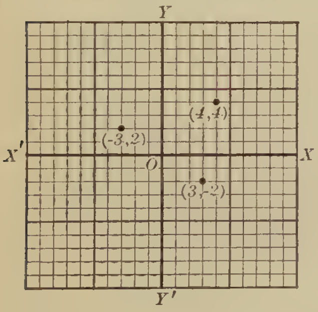


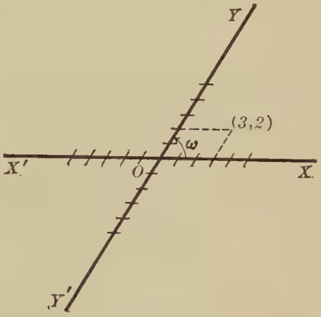
4. Notation. — In naming a point by its coördinates we write them in a parenthesis, putting the abscissa first.

Thus in the figure P is the point $(4, -3)$. If the coördinates are variable or unknown, the abscissa is denoted by the letter x, and the ordinate by y. Fixed points of which the coördinates are not known or are arbitrary, will be distinguished by means of subscripts, being lettered P_1, P_2, etc., and represented by the coördinates (x_1, y_1), (x_2, y_2), etc.

5. Plotting of Points. — It is clear that any point in a plane is fixed by means of its coördinates; for the abscissa locates it on a parallel to the y-axis and the ordinate on a parallel to the x-axis, and these meet in one point.

To locate a point whose coördinates are given (or as commonly stated, to *plot* the point), first mark the coördinate axes and choose a convenient unit of measure, then measure off the abscissa on the x-axis and the ordinate from the end of the abscissa. Thus to plot $(-3, 2)$ count 3 units to the left on the x-axis and 2 units up. This is especially convenient when using coördinate paper, *i.e.*, paper ruled with sets of equally spaced parallel lines.





The system of coördinates which has been described is the rectangular system and is the particular case of Cartesian coördinates which is usually employed. In the general Cartesian system the axes are not necessarily perpendicular, but may be *oblique*. All of the above definitions hold for the oblique system, which is illustrated above.

PROBLEMS

1. Plot each of the following groups of points, using a single set of coördinate axes:

(*a*) $(2, 5), (3, 0), (-2, 5)$;
(*b*) $(-3, 0), (5, -8), (1, 12)$;
(*c*) $(5, -12), (-13, -6), (-2.5, 6.6)$;
(*d*) $(10, 0), (0, -8), (8.5, 7.5)$;
(*e*) $(2, -6), (-3, -3), (0, -\sqrt{7})$;
(*f*) $(-1, 2), (-2, -3), (-\sqrt{3}, 0)$.

2. Draw the triangle whose vertices are as follows:

(*a*) $(3, -1), (-2, 5), (-8, -4)$;
(*b*) $(10, 6), (2, -2), (16, -8)$;
(*c*) $(0, 0), (-8, 8), (4, 6)$;
(*d*) $(2, 4), (4, \sqrt{8}), (3, 11)$;
(*e*) $(5, 3), (5 + 4\sqrt{2}, 3 + 4\sqrt{2}), (5 + 8\sqrt{2}, 3 - \sqrt{2})$.

3. Draw the quadrilateral whose vertices are as follows:

(*a*) $(-2, 0), (2, 4), (6, 0), (2, -4)$;
(*b*) $(0, -6), (8, 2), (0, 6), (-8, 2)$;
(*c*) $(0, 0), (5, 3), (0, 8), (5, 11)$.

4. Draw the triangle with the given vertices and find its area:

(*a*) $(8, 6), (8, -4), (10, -4)$;
(*b*) $(-2, 1), (-2, 9), (3, 9)$;
(*c*) $(-3, -1), (-3, -9), (7, -9)$;
(*d*) $(0, 0), (0, 8), (-6, 8)$.

5. Draw the triangle with the given vertices and find its area:

(*a*) $(7, -5), (7, -13), (-2, -13)$;
(*b*) $(-1, -5), (-1, 8), (3, 8)$;

(*c*) $(8, 8)$, $(16, 8)$, $(16, -4)$
(*d*) (a, b), (a, c), (d, c).

6. What is the length of the hypotenuse of each triangle in Problem 4?

7. What is the length of the hypotenuse of each triangle in Problem 5?

8. Find the coördinates of the point 8 units to the right and 10 units below the point $(-2, 4)$.

9. Construct with compasses on coördinate paper $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and estimate the value of each.

Hint. — Each radical will be the measure of the hypotenuse or leg of a right triangle.

10. Draw a circle of radius 10 with its center at the origin. Draw radii at 20° intervals up to 180°. Estimate from the figure the coördinates of the ends of these radii.

11. From the figure of Problem 10 make a table of sines and cosines of the angles drawn.

12. What is the abscissa of all points on the y-axis? the ordinate of all points on the x-axis? What are the coördinates of the origin?

13. To what quadrants is a point limited if its abscissa and ordinate have like signs? unlike signs?

14. If a point moves on a parallel to the x-axis, which of its coördinates remains constant? which if it moves on a perpendicular to the x-axis?

15. (*a*) What is the locus of a point whose abscissa is 6? whose ordinate is -6?

(*b*) What is the locus of all points having the same abscissa? having the same ordinate?

16. What is the locus of points whose abscissas and ordinates are (*a*) equal? (*b*) numerically equal, but of unlike sign? Why?

17. A square whose side is of length a has its vertices on the coördinate axes. Find the coördinates of the vertices.

18. A square whose side is of length $2a$ has its center at the origin and its sides parallel to the axes. Find the coördinates of its vertices.

19. An equilateral triangle of side a has one vertex at the origin and the opposite side parallel to the y-axis. Find the coördinates of the other vertices.

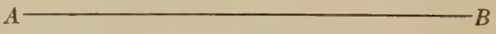
20. The base of an equilateral triangle extends from $(-a, 0)$ to $(a, 0)$. What are the coördinates of the vertex?

21. A rhombus has one angle of 60° and two vertices at $(0, 0)$ and $(a, 0)$. Find the coördinates of the other vertices if (a) both are in the first quadrant; (b) one is in the second quadrant.

22. A regular hexagon of side a is placed so that one diagonal lies along the x-axis and the center is at the origin. Find the coördinates of the vertices.

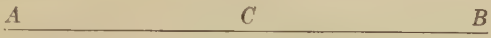
23. Three vertices of a parallelogram are $(0, 0)$, $(a, 0)$, and (b, c). What is the fourth vertex?

6. Directed Lines. — We have referred to the advantage of using positive and negative lines in trigonometry and have defined the signs of coördinates. In analytic geometry we constantly use *directed lines*, that is, lines whose lengths are reckoned as positive or negative according to the direction in which they are read. For example, if the positive direction is from left to right, and AB is 8 units long, then



$AB = 8$, while $BA = -8$. Thus, *changing the direction of reading a line changes its sign, i.e.,* $BA = -AB$.

In adding or subtracting line segments great care should be taken to avoid errors of sign. It is advisable for beginners to read all segments in one direction in performing such operations, later reversing segments as required. For example, let us find the relation between AB, BC, and AC in the figure.



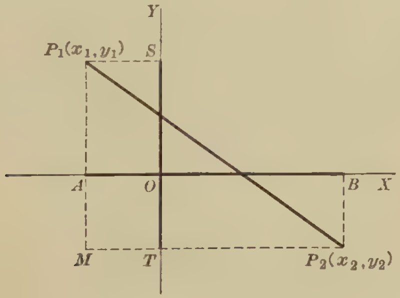
We have $\quad AB = AC + CB = AC - BC.$

In the Cartesian coördinate system *the positive direction on all lines parallel to the x-axis is to the right; on lines not parallel to the x-axis the positive direction is upward.* All lines should be read in the positive direction unless it is intended that their lengths are to be considered negative. Since coördinates are the measures of directed lines, they must be read up or to the right if positive, and down or to the left if negative.

Exercise 1. Prove that if A and B are any two points on a directed line passing through the point O,

$$AB = OB - OA. \qquad \text{(Three cases.)}$$

7. Projections of a Directed Segment on the Coördinate Axes. — THEOREM. — *The projection on the x-axis or any line parallel to it of the segment P_1P_2 is $x_2 - x_1$. The projection on the y-axis or any line parallel to it of the segment P_1P_2 is $y_2 - y_1$.*



PROOF. In the figure P_1P_2 is the given (directed) segment, and MP_1 and P_2B are parallel to the y-axis while P_1S and MP_2 are parallel to the x-axis.

The projection on the x-axis is

$$AB = OB + AO = OB - OA = TP_2 - SP_1.$$

By the definition of abscissa $TP_2 = x_2$ and $SP_1 = x_1$. Substituting these values above we have

$$AB = x_2 - x_1.$$

In the same way the projection on the y-axis is

$$ST = P_1M = BP_2 + P_1A = y_2 - y_1.$$

Since the x-axis is usually drawn horizontally it will be convenient to refer to these projections as the *horizontal* and *vertical projections.* Since this theorem is used constantly the proof should be worked out for several positions of the points P_1 and P_2 so as to show the validity of the formulas for all possible positions. Note that the horizontal projection of P_2P_1 is $x_1 - x_2$, and that its vertical projection is $y_1 - y_2$.

Exercise 2. Prove the theorem of § 7 when

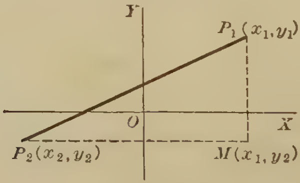
(*a*) P_1 lies in the fourth quadrant and P_2 in the second;

(*b*) P_1 lies in the second quadrant and P_2 in the third;

(*c*) P_1 lies in the third quadrant and P_2 in the fourth.

8. Distance Formula. — *The distance between any two points,* P_1 (x_1, y_1) *and* P_2 (x_2, y_2), *is given by the formula,*

$$\boldsymbol{d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.} \tag{1}$$



For $$d = P_2P_1 = \sqrt{\overline{P_2M}^2 + \overline{MP_1}^2}.$$

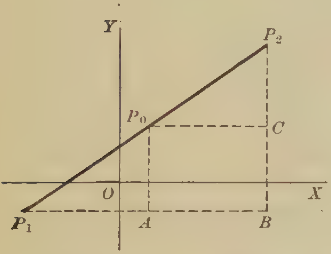
But, by § 7, $P_2M = x_1 - x_2$,

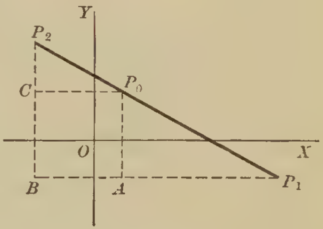
and $MP_1 = y_1 - y_2$.

Substituting, we have the formula.

9. Point of Division Formulas. — *The coördinates* (x_0, y_0) *of a point* P_0 *dividing a line joining* $P_1(x_1, y_1)$ *and* $P_2(x_2, y_2)$ *in a given ratio* $r_1 : r_2$ *are given by the formulas*

$$x_0 = \frac{r_2x_1 + r_1x_2}{r_1 + r_2}, \qquad y_0 = \frac{r_2y_1 + r_1y_2}{r_1 + r_2}. \tag{2}$$





Since parallel lines cut off proportional segments on two transversals, we have

$$P_1P_0 : P_0P_2 = P_1A : P_0C.$$

But $P_1P_0 : P_0P_2 = r_1 : r_2$ by hypothesis. Also, by § 7, $P_1A = x_0 - x_1$, and $P_0C = x_2 - x_0$.

Substituting, $$\frac{r_1}{r_2} = \frac{x_0 - x_1}{x_2 - x_0}.$$

Solving for x_0, $$x_0 = \frac{r_2x_1 + r_1x_2}{r_1 + r_2}$$

We derive the formula for y_0 in the same manner.

When P_0 is the mid-point of the line, $r_1 = r_2$ and these formulas become:

$$x_0 = \tfrac{1}{2}(x_1 + x_2), \qquad y_0 = \tfrac{1}{2}(y_1 + y_2), \tag{2 a}$$

which are called the *mid-point formulas.*

Exercise 3. State in words the rule expressed by (*a*) the distance formula, (*b*) the mid-point formulas.

Exercise 4. Derive the mid-point formulas directly from a figure.

Example. — A segment joins $(-3, 6)$ to $(13, 30)$. (*a*) Find the point on this segment which is $\frac{5}{8}$ the distance from the first to the second point. (*b*) To what point must the segment be extended beyond $(13, 30)$ in order to increase its length by one fourth?

Solution. — Let P_1 be $(-3, 6)$ and P_2 be $(13, 30)$. (*a*) We must find $P_0\,(x_0, y_0)$ so that $P_1P_0 = \frac{5}{8}P_1P_2$. If this relation exists, we see at once that $P_1P_0 : P_0P_2 = 5 : 3$.

Then, in Formula 2, $r_1 = 5$, $r_2 = 3$, $x_1 = -3$, $y_1 = 6$, $x_2 = 13$, and $y_2 = 30$. Substituting, we have

$$x_0 = \frac{(5)\,(13) + (3)\,(-3)}{5+3} = 7$$

and

$$y_0 = \frac{(5)\,(30) + (3)\,(6)}{5+3} = 21$$

Thus the first required point is $(7, 21)$.

(*b*) If, in this case, P_0 denotes the end of the extended segment, we have $P_1P_0 = \frac{5}{4}P_1P_2$. But since P_0 lies without P_1P_2, P_1P_0 and P_0P_2 are of unlike signs. Hence

$$\frac{P_1P_0}{P_0P_2} = \frac{5}{-1} = \frac{-5}{1}.$$

Taking $r_1 = 5$ and $r_2 = -1$, we find from the formula that

$$x_0 = \frac{(5)\,(13) + (-1)\,(-3)}{5-1} = 17$$

and

$$y_0 = \frac{(5)\,(30) + (-1)\,(6)}{5-1} = 36.$$

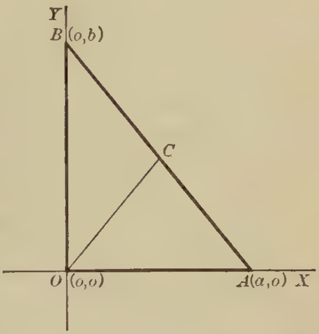
Hence the required point is $(17, 36)$. The same result is obtained if we take $r_1 = -5$ and $r_2 = 1$.

10. Applications to Geometric Proofs. — By means of these formulas and others which will follow, many of the theorems of plane geometry can be proved in a very simple manner.

Example. — Show by analytic means that the mid-point of the hypotenuse of any right triangle is equidistant from the vertices.

Solution. — Let the lengths of the legs be a and b. Let the coördinate axes be taken along the legs of the triangle. Then the vertices are $(0, 0)$, $(a, 0)$, $(0, b)$. Let C be the mid-point of the hypotenuse. By Formula $2a$ its coördinates are $\left(\frac{a}{2}, \frac{b}{2}\right)$. The distance formula gives

$$OC = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \tfrac{1}{2}\sqrt{a^2 + b^2}.$$



The same result is obtained for CB and AC, which proves the theorem.

Note. — In this problem the lengths of the sides were expressed by letters so as to insure that OBA should represent *any* right triangle. This procedure must be followed in all such problems in order to make the proof general and not merely a verification of a special case. The particular choice of coördinate axes obviously does not affect the size or shape of a figure.

PROBLEMS

1. Show that the triangle whose vertices are $(-4, 16)$, $(2, 6)$, and $(12, 12)$ is isosceles.

Solution. — Using the distance formula, we let P_1 be $(-4, 16)$ and P_2 be $(2, 6)$. Then $x_1 = -4$, $y_1 = 16$, $x_2 = 2$, and $y_2 = 6$. Substituting, we have

$$d_1 = \sqrt{(-4-2)^2 + (16-6)^2} = \sqrt{136} = 2\sqrt{34}.$$

Applying the same formula to the points $(-4, 16)$ and $(12, 12)$, we find

$$d_2 = \sqrt{(-4-12)^2+(16-12)^2} = \sqrt{272} = 4\sqrt{17}.$$

Similarly the distance between $(2, 6)$ and $(12, 12)$ is

$$d_3 = \sqrt{(2-12)^2+(6-12)^2} = \sqrt{136} = 2\sqrt{34}.$$

Since $d_1 = d_3$, the triangle is isosceles.

In solving problems a figure should always be drawn, because it often suggests a means of solving the problem or of shortening the solution and it always furnishes a check on the accuracy of arithmetical work. In this problem a figure would show that d_2 cannot possibly be equal to either of the other sides and hence it was unnecessary to find its length.

2. Find the lengths of the sides of the triangles whose vertices are

(*a*) $(4, 3)$, $(2, 2)$, $(-3, 5)$;
(*b*) $(-3, 1)$, $(7, -2)$, $(-6, 5)$;
(*c*) $(5, 4)$, $(-3, 2)$, $(-3, -6)$;
(*d*) $(4, 5)$, $(2, -3)$, $(-6, -3)$;
(*e*) $(8, 10)$, $(-7, -6)$, $(12, -20)$;
(*f*) $(a, 0)$, $(-a, 0)$, $(0, b)$.

3. Find the perimeter of each of the triangles in Problem 2.

4. Show that the following are the vertices of isosceles triangles:

(*a*) $(2, 4)$, $(5, 1)$, $(6, 5)$;
(*b*) $(2, 6)$, $(6, 2)$, $(-3, -3)$;
(*c*) $(-6, 5)$, $(8, 2)$, $(-3, -9)$.

5. Show that $(3, 3)$, $(-3, -3)$, $(3\sqrt{3}, -3\sqrt{3})$ are the vertices of an equilateral triangle, and find the length of its medians.

6. Show that $(3, 1)$, $(6, 5)$, $(-1, 4)$ are the vertices of a right triangle. What is its area and the distance between the midpoints of its legs?

7. Find the area of the triangle whose vertices are $(-5, -3)$, $(7, -8)$, and $(12, 4)$.

8. Show that the points (3, 4), $(2, 1 + \sqrt{12})$, (4, 3) lie on a circle with its center at (1, 1)

9. Show that (15, 8), (23, −7), (8, −15), (0, 0) are the vertices of a square. Find the length of its diagonals.

10. Find the coördinates of the point dividing in the ratio 3 : 5 the segment whose extremities are:

(*a*) (5, 6) and (13, −10);
(*b*) (−1, 5) and (3, −9);
(*c*) (−6, 3) and (8, −7);
(*d*) (0, 0) and (−24, 6).

11. In each of the following find the coördinates of the point three fourths of the distance from the first to the second point:

(*a*) (5, 6) and (13, −8);
(*b*) (−1, 5) and (3, −4);
(*c*) (−6, 8) and (3, −2);
(*d*) (0, 0) and (*a*, *b*).

Hint. — First determine the ratio of the segments.

12. Find four points on the line passing through (−4, 7) and (8, −3) each of which is twice as far from one of these points as from the other.

13. Find the point of intersection of the medians of the triangles whose vertices are the points in Problem 2.

Hint. — We know from plane geometry that it is two thirds of the distance from a vertex to the middle of the opposite side.

14. Find a point 10 units distant from the point (−3, 6) and with the abscissa 3.

15. Find a point equidistant from the points (13, 8), (6, 15) and (−4, −9).

16. Verify the mid-point formulas by means of the distance formula.

17. Two of the vertices of an equilateral triangle are (1, 4) and (3, −2). Find the other vertex.

18. Prove analytically that:

(*a*) the diagonals of a rectangle are equal;

(*b*) the diagonals of an isosceles trapezoid are equal;

(*c*) a line joining the middle points of two sides of a triangle is one half the third side;

(*d*) the median of a trapezoid is one half the sum of the bases;

(*e*) the lines joining the mid-points of opposite sides of a quadrilateral bisect each other.

19. Prove analytically that in any triangle:

(*a*) the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides decreased by twice the product of one of those sides and the projection of the other upon it;

(*b*) the sum of the squares of two sides is equal to twice the square of one half the third side, increased by twice the square of the median on that side;

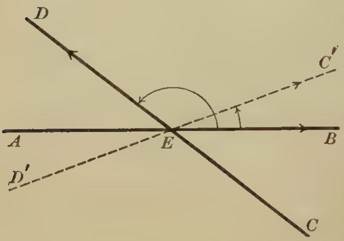
(*c*) the sum of the squares of the medians is equal to three fourths the sum of the squares of the three sides.

20. Prove analytically that:

(*a*) the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals;

(*b*) the sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals increased by four times the square of the line joining the mid-points of the diagonals.

11. Angle between two Lines. — *The angle of intersection of two directed lines is understood to be the angle between their positive directions; i.e.,* if two lines intersect, their angle of intersection is that one of the four angles formed for which the sides are positive *if read from the point of intersection.* Thus, in the figure the angle of intersection of AB



and CD is BED. The object of this convention is to avoid ambiguity and its effect is to keep the value of any angle between 0° and 180°. For example, suppose the line CD to be rotated around E so that the angle BED increases from 0° until ED reaches and passes the position EA. After that the positive direction on CD becomes EC' and the angle between the lines is BEC'.

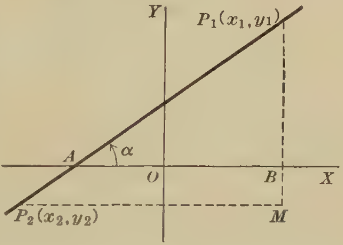
12. Inclination and Slope. — *The inclination of a line is its angle of intersection with the x-axis.* If it is parallel to the x-axis, its inclination is zero.

The slope of a line is the tangent of its inclination. We denote the inclination by the letter α, the slope by m. Thus $m = \tan \alpha$. When m is positive, α is acute and the line extends *upward* to the right; when m is negative, α is obtuse, and the line extends *downward* to the right.

The slope fixes the direction of a line. Hence a line is determined when its slope and one point on it are known.

13. The Slope Formula. — *The slope of a line passing through the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula*

$$m = \frac{y_1 - y_2}{x_1 - x_2} \qquad (3)$$



Take the case where P_1 lies in the first quadrant and P_2 in the third.

Then $$\tan \alpha = \frac{BP_1}{AB} = \frac{MP_1}{P_2M}.$$

Now, by § 7, $$MP_1 = y_1 - y_2,$$

and $$P_2M = x_1 - x_2.$$

Substituting, we have the formula.

Exercise 5. Derive the slope formula when

(*a*) P_1 is in the fourth quadrant and P_2 in the second;

(*b*) P_1 is in the second quadrant and P_2 in the third;

(*c*) P_1 is in the third quadrant and P_2 in the fourth;

(*d*) both points are in the first quadrant and the inclination is greater than 90°.

Exercise 6. State in words the rule expressed by the slope formula.

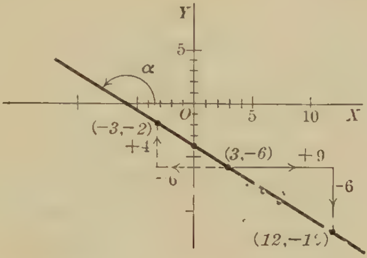
14. Applications of the Slope Formula. — 1. *To find the slope and inclination of a line through two given points.* To solve this problem we simply substitute the coördinates of the points in the formula and look up the corresponding inclination. For example, consider the line passing through $(-5, 6)$ and $(11, -6)$. Formula 3 gives at once

$$m = \frac{6+6}{-5-11} = -\tfrac{3}{4} = -0.7500.$$

The inclination is the angle whose tangent is -0.7500. We recall from trigonometry that this is the supplement of the angle whose tangent is $+0.7500$, which is $36°52'$. Hence $\alpha = 180° - 36°52' = 143°8'$.

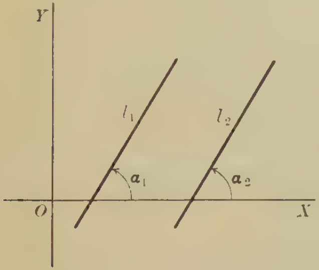
2. *To draw a line through a given point with a given slope.* The natural procedure would be to look up the inclination in the tables and draw the line with a protractor. It is easier, however, to use the slope formula. For example, consider the line through $(3, -6)$ with slope $-\frac{2}{3}$. Since the slope formula shows that the slope is merely the difference of the ordinates divided by the difference of the abscissas, another point on the line is that 3 units to the right and 2 units below $(3, -6)$, namely $(6, -8)$. As $-\frac{2}{3} = -\frac{4}{6} = -\frac{6}{9}$, etc., we also get points on the line by counting 6 units to the right and 4 down,

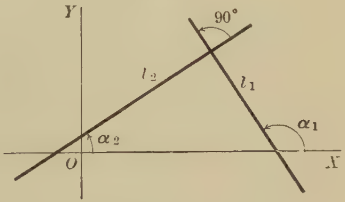
or 9 to the right and 6 down, etc. We may also count 6 to the left and 4 up, etc.



15. Parallelism and Perpendicularity. — *If two lines are parallel their slopes are equal, and conversely; if they are perpendicular their slopes are negative reciprocals, and conversely.*

If l_1 and l_2 are parallel, obviously $\alpha_1 = \alpha_2$, whence the slopes are equal.





If l_1 is perpendicular to l_2, we have $\alpha_1 = \alpha_2 + 90°$. Since $\tan(\alpha + 90°) = -\cot\alpha$ by trigonometry, we have

$$\tan\alpha_1 = \tan(\alpha_2 + 90°) = -\cot\alpha_2 = -\frac{1}{\tan\alpha_2}.$$

Substitution of m_1 and m_2 gives

$$m_1 = -\frac{1}{m_2}, \qquad \text{or} \qquad m_1 m_2 = -1.$$

Let the student prove the converse theorems.

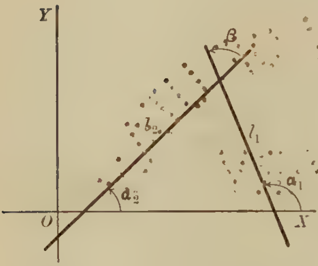
Summarizing, we have:

Test for parallelism,

$$m_1 = m_2; \tag{4}$$

Test for perpendicularity,

$$m_1 m_2 = -1. \tag{5}$$



16. The Angle Formula. — *The angle β between two lines is given by the formula*

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2}, \tag{6}$$

m_1 denoting the slope of the line of greater inclination.

Obviously $\alpha_1 = \alpha_2 + \beta$ and $\beta = \alpha_1 - \alpha_2$, whence

$$\tan \beta = \tan (\alpha_1 - \alpha_2) = \frac{\tan \alpha_1 - \tan \alpha_2}{1 + \tan \alpha_1 \tan \alpha_2}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2}.$$

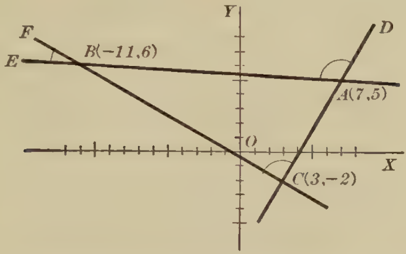
Note. — When the inclinations are known, it is obvious from the above that $\beta = \alpha_1 - \alpha_2$ may be used as an alternative to Formula 6.

Exercise 7. Derive the angle formula for the case when the point of intersection is below the x-axis.

Example. — Find the angles of the triangle whose vertices are A (7, 5), B (−11, 6), and C (3, −2).

Solution. — The triangle is illustrated in the figure. By means of the slope formula we find at once that the slopes of AB, CB, and CA are respectively $-\frac{1}{18}$, $-\frac{4}{7}$, and $\frac{7}{4}$. Since the product of the last two is −1, the angle BCA is 90° by Formula 5.

At A the angle given by Formula 6 is DAB, since the positive direction from A on the side CA is AD and not AC. The inclination of AD is acute and that of AB is obtuse, hence we set $m_1 = \text{slope } AB = -\frac{1}{18}$ and $m_2 = \frac{7}{4}$. Then Formula 6 gives



$$\tan DAB = \tan\beta = \frac{-\frac{1}{18} - \frac{7}{4}}{1 - \frac{7}{72}} = -2.$$

The tables give $DAB = 180° - 63°26'$, whence $BAC = 63°26'$.

Since ABC is a right triangle, angle ABC is the complement of angle BAC. By finding it directly, however, we can use this fact as a check on our work.

At the vertex B angle "β" of Formula 6 is the angle EBF, which is equal to the required angle ABC. The larger inclination is that of AB; hence $m_1 = \text{slope } AB = -\frac{1}{18}$ and $m_2 = -\frac{4}{7}$. Therefore

$$\tan ABC = \frac{-\frac{1}{18} + \frac{4}{7}}{1 + \frac{4}{126}} = \tfrac{1}{2} = 0.5000.$$

Hence $$ABC = 26°34'.$$

Check: $$ABC + BCA + CAB = 180°.$$

PROBLEMS

1. Find the slope and the inclination of the line passing through

(*a*) $(2, 3), (12, 18)$;
(*b*) $(3, 8), (-6, -6)$;
(*c*) $(-1.2, 6.3), (9, -6)$;
(*d*) $(-5, 8), (-18, -7)$;
(*e*) $(\sqrt{2}, \sqrt{3}), (-\sqrt{3}, \sqrt{2})$;
(*f*) $(a + b, c), (a, b + c)$.

2. Find the slope and the inclination of the line passing through

(*a*) $(3, 3), (-3, -3)$;
(*b*) $(-9, 6), (-12, 9)$;
(*c*) $(5, 0), (6, \sqrt{3})$;
(*d*) $(-6, 0), (-4, 2\sqrt{3})$;
(*e*) $(0, -8), (-6, 2)$;
(*f*) $(0, 0), (-4\sqrt{3}, -20)$.

3. Prove by means of slopes that $(2, 3)$, $(6, -3)$, and $(-2, 9)$ are on the same straight line.

4. Prove that $(4, 7)$ is on the line joining the points $(6, 6)$ and $(2, 8)$ and is equidistant from them.

5. Prove that the line joining $(6, -3)$ and $(2, 8)$ is perpendicular to the one joining $(0, 0)$ and $(11, 4)$.

6. What is the inclination of a line parallel to the y-axis? perpendicular to the y-axis? What are the slopes of these lines?

7. Find the angle between the line joining $(-6, -3)$ and $(2, 7)$ and each of the lines in Problem 2.

8. Find the angle between the line joining $(1, 7)$ and $(6, -2)$ and each of the lines in Problem 1.

9. Find the angle between a line whose inclination is 135° and each of the lines in Problem 2.

10. Find the angle between a line whose inclination is 40° and each of the lines in Problem 1.

11. Find the angle between the line from $(8, -4)$ to $(-6, 12)$ and the line bisecting the first quadrant.

12. Prove by means of slopes that the following points are vertices of a right triangle and find its acute angles:

(*a*) $(10, 0), (16, 8), (2, 6)$;
(*b*) $(5, 5), (2, 8), (10, 10)$;
(*c*) $(1, 1), (9, -5), (-5, -7)$;
(*d*) $(0, 0), (10, 6), (13, 1)$.

13. Prove by means of slopes that the following points are the vertices of a parallelogram and find its angles:

(*a*) $(-1, -2), (0, 1), (3, 4), (2, 1)$;
(*b*) $(3, -3), (4, 0), (7, 3), (6, 0)$;
(*c*) $(-5, -4), (1, 4), (11, 16), (5, 8)$;
(*d*) $(3, -1), (8, 8), (11, 1), (6, -8)$.

14. Prove that $(0, -2)$, $(4, 2)$, $(0, 6)$, and $(-4, 2)$ are the vertices of a square.

15. Find the angles of the triangle whose vertices are:

(*a*) $(-2, 2), (4, 2), (1, 5)$;
(*b*) $(1, 1), (-1, -1), (5, -5)$;
(*c*) $(5, 8), (8, -6), (-8, -3)$;
(*d*) $(-6, 6), (10, 4), (2, 2)$.

16. Prove analytically that

(*a*) the diagonals of any square are perpendicular;

(*b*) the median of a trapezoid is parallel to the bases;

(*c*) the lines joining the mid-points of the sides of any quadrilateral form a parallelogram;

(*d*) the lines joining the mid-points of the sides of a rectangle form a rhombus.

17. Show that two lines for which $m_1 = -m_2$ form with the x-axis an isosceles triangle.

18. Show that the points (a, b), $(a + d, b + e)$, $(a + 2d, b + 2e)$, $(a + 3d, b + 3e)$, etc., all lie on the same straight line and find its slope.

19. Derive formula (5) from formula (6).

20. Three vertices of a parallelogram are in order $(8, 6)$, $(2, 5)$, and $(4, -2)$. Find the fourth vertex.

21. Find a point on the line joining $(2, 3)$ and $(-6, 7)$ and 2 units distant from the former point.

CHAPTER II

CURVES AND EQUATIONS

17. Introduction. — The coördinates x, y, if no restriction is placed on their values, represent any point in the plane. If, however, the values of x and y are subject to certain conditions, points having these coördinates will lie upon certain lines or curves. For example, if y is unrestricted, but x always equals -6, all such points will lie upon a line parallel to the y-axis and 6 units to the left. Again, the locus of all points whose abscissas and ordinates are equal is the bisector of the first and third quadrants. Such restrictions upon coördinates are expressed by means of equations. Thus, the equation of the first locus is $x = -6$, of the second $x = y$. As a general definition, we have:

The equation of a locus is an equation satisfied by the coördinates of all points on the given locus, and conversely. The curve which contains all points whose coördinates satisfy the given equation and no other points is called the *locus* or *graph* of the equation.

The derivation of equations of loci and the study of graphs by means of their equations form the chief part of elementary Analytic Geometry.

18. Plotting of Graphs. — When a locus is given by its equation, the shape of the curve is sometimes evident from the form of the equation, as in examples mentioned above, and as we shall see in more complicated problems later. Usually the graph is constructed by a process called plotting the graph of the equation, in which we proceed as follows:

Solve the equation for y in terms of x.

Set x equal to convenient positive and negative values (generally integral) *and compute the corresponding values of y.* Each pair of values of x and y is a solution of the equation, and hence the point of which these are the coördinates lies on the curve by the definition of the locus.

Make a table of values by arranging these pairs in order according to the magnitudes of the values of x.

Plot the points thus tabulated and join them by a smooth curve in the order of the table. This gives an approximation to the true curve which becomes more exact when a larger number of points is plotted.

REMARKS. — It is sometimes more convenient to solve for x in terms of y. In this case the above is applicable on interchanging x and y.

If the solution for y in terms of x involves a square root, the double sign must be used with the radical and in general two points will be found for each value of x. A similar remark applies to a solution for x in terms of y.

The scale on which the curve is drawn must be small enough so that the tabulated points will fall within the limits of the paper, but it should not be chosen so small that the drawing cannot be well done.

Sometimes the values of either x or y will be so large that it is impossible to draw a smooth curve through the plotted points. In this case use one scale for x and another for y. This should not be done unless necessary, for the figure thus drawn will not be a true image of the locus, but will be a distortion much like those produced by convex mirrors.

When the points plotted do not show the form of the curve, look for an error in the table of values. If there is no error, assume intermediate fractional values of x or y and plot the corresponding points.

The table of values should be extensive enough to show the form of the curve completely. If it extends to infinity, the plot should contain all parts with considerable curvature and should extend far enough to show the direction of the curve beyond the limits of the paper.

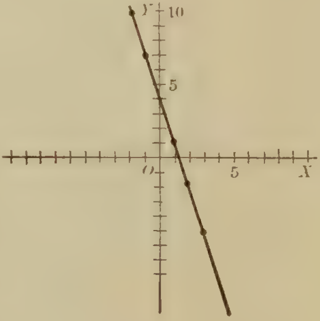
Example 1. — Plot the graph of $3x + y - 4 = 0$.

x	y
-2	10
-1	7
0	4
1	1
2	-2
3	-5

Solving for y,

$$y = -3x + 4.$$

Substituting $x = -2, -1, 0, 1, 2, 3$, we get $y = 10, 7, 4, 1, -2, -5$ respectively. Thus points on the line are $(-2, 10)$, $(-1, 7)$, $(0, 4)$, etc. The table of values is at the left and the plot at the right.



Example 2. — Plot the graph of $5y^2 - 10y = x - 15$.

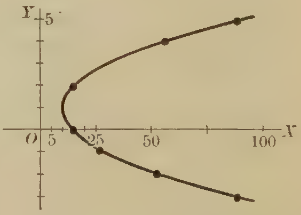
y	x
-3	90
-2	55
-1	30
0	15
1	10
2	15
3	30
4	55
5	90

Here it is easier to solve for x in terms of y:

$$x = 5y^2 - 10y + 15.$$

In the plot each unit on the y-axis has been taken equal to 10 units on the x-axis. This is because the values of x are so much larger than those of y that if the same scale were used on both axes, the curve would be impossible to draw on a small sheet of paper.

In such cases the scale used should be indicated on the graph.



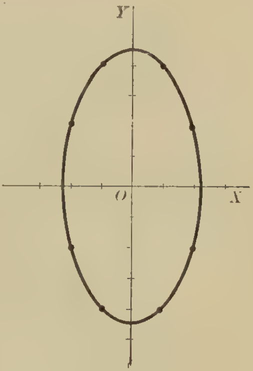
EXAMPLE 3. — Plot the graph of $4x^2 + y^2 = 20$.

x	y
-3	imag.
-2	± 2
-1	± 4
0	$\pm 2\sqrt{5}$ $= \pm 4.5$
1	± 4
2	± 2
3	imag.

Solving for y in terms of x, we get:

$$y = \pm\sqrt{20 - 4x^2}$$
$$= \pm 2\sqrt{5 - x^2}.$$

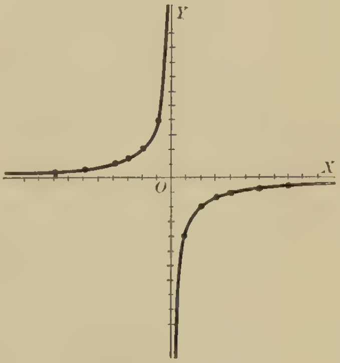
Here no part of the curve can lie outside of $x = +3$ and $x = -3$, but there are two points corresponding to each of the values $x = \pm 2$. To find out where the curve crosses the x-axis, we set $y = 0$ in the equation and solve, getting $x = \pm\sqrt{5} = \pm 2.2^+$. Adding the points $(2.2^+, 0)$ and $(-2.2^+, 0)$ to the plot, we are able to draw the entire curve.



EXAMPLE 4. — Plot the graph of $xy = -4$, or $y = -\frac{4}{x}$.

x	y
1	-4
2	-2
3	$-\frac{4}{3}$
4	-1
6	$-\frac{2}{3}$
8	$-\frac{1}{2}$

x	y
-1	4
-2	2
-3	$\frac{4}{3}$
-4	1
-6	$\frac{2}{3}$
-8	$\frac{1}{2}$



Since x cannot equal 0,[*] we must take fractional values of x between ± 1 in order to find the shape of the curve. These give additional points:

x	y
$\frac{1}{2}$	-8
$\frac{1}{4}$	-16

x	y
$-\frac{1}{2}$	8
$-\frac{1}{4}$	16

[*] This is due to the fact that the substitution of $x = 0$ in the equation involves division by 0, an impossible operation. Since

Adding these to the plot, we see that the curve goes to infinity along the x- and y-axes in the second and fourth quadrants.

19. Equations Containing Arbitrary Constants. — Consider the equation $4x^2 + y^2 = a^2$. For each value of a this equation has a locus. When $a = \sqrt{20}$, we have the closed curve of Example 3 above; when $a = 5$, we have a similar curve, somewhat larger than the one in the figure, etc. The letter a is called an *arbitrary* constant, since real values may be assigned to it at pleasure. As we shall see later, the loci of equations of a given form constitute a class or *system* of curves having many properties in common and frequently of the same general shape. Thus the graph of any equation of the form $4x^2 + y^2 = a^2$ is an ellipse twice as long as it is wide.

In plotting such an equation we set the arbitrary constant equal to a convenient value and proceed as in § 18. The value of the arbitrary constant chosen should not be zero, as this value will generally not show the properties of the system. For a similar reason, if there are two arbitrary constants, the same value should not be used for both.

PROBLEMS

Plot the graphs of the following equations:

1. $x = 6$.

2. $y = 0$.

3. $y = -2$.

4. $3y = x$.

5. $y = -2x$.

6. $3x + 4y = 0$.

7. $4x - 3y = 24$.

8. $x + y = 10$.

9. $3x - y = 12$.

10. $3x - y = 21$.

11. $4y = 64 - x^2$.

12. $y = x^2 - 8x + 12$.

13. $2y = x^2 - 4x + 10$.

14. $y^2 = 8x$.

y becomes indefinitely great as x *approaches* 0, it is sometimes stated that for $x = 0$, $y = \infty$, but this abbreviation should not be allowed to obscure the fact that for $x = 0$ there is *no* point on the curve.

15. $y = x^2$.
16. $x^2 - 3x + 2y = 6$.
17. $x = y^2 - 6y + 5$.
18. $y^2 - y = x + 2$.
19. $x^2 + y^2 = 64$.
20. $x^2 + y^2 = 25$.
21. $x^2 + 4y^2 = 64$.
22. $9x^2 + y^2 = 144$.
23. $4x^2 + y^2 = 36$.
24. $25x^2 + 16y^2 = 400$.
25. $x^2 + 2y^2 = 2$.
26. $x^2 - y^2 = 9$.
27. $x^2 - y^2 + 16 = 0$.
28. $y^2 = 25 + x^2$.
29. $x^2 - 9y^2 = 36$.
30. $9x^2 - y^2 + 9 = 0$.
31. $x^2 + 4x + 4 = y^2 - 3y$.
32. $y = x^3 - 9x^2$.
33. $y = x^3$.
34. $y = x^3 - 7x^2 - 8x - 12$.
35. $10y = x^3 - 16x$.
36. $4y = x^4 - 8x^2$.
37. $8y^2 = x^3$.
38. $y^3 = x^2$.
39. $xy = 48$.
40. $2xy = -15$.
41. $xy = y + 2$.
42. $xy = 3x - 1$.
43. $xy^2 = 16$.
44. $x^2y = 25$.
45. $xy - y^2 = 10$. (Solve for x.)
46. $x + y - 2xy = 0$.

47. Find two of the above curves which pass through
(*a*) $(0, 0)$; (*b*) $(-1, 1)$; (*c*) $(1, 1)$.

48. $y = mx$.
49. $y = ax^2$. $(a > 0.)$
50. $y = ax^2$. $(a < 0.)$
51. $y^2 = ax$. $(a > 0.)$
52. $y^2 = ax$. $(a < 0.)$
53. $y = ax^3$. $(a > 0.)$
54. $y = ax^3$. $(a < 0.)$
55. $y^2 = ax^3$.
56. $x^2 + y^2 = a^2$.
57. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
58. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
59. $xy = a$. $(a > 0.)$
60. $xy = a$. $(a < 0.)$
61. $y = x^3 - a^2x$.
62. $x^2y = a^2$.
63. $y = x^3 - ax^2$.

20. Derivation of Equations. — When a curve is not given by means of its equation, it is usually defined as the locus of a point satisfying certain definite conditions. Thus the circumference of a circle is the locus of a point at a given distance (the radius) from a fixed point (the center). When a curve is defined in this way we have the problem of finding its equation.

This problem is to translate the definition of the curve into algebraic language and to obtain as a result an equation in x and y *satisfied by the coördinates of every point on the locus and by those of no other point.* No rule can be stated which will automatically yield a solution of every such problem, but the following general directions should be followed as far as practicable.

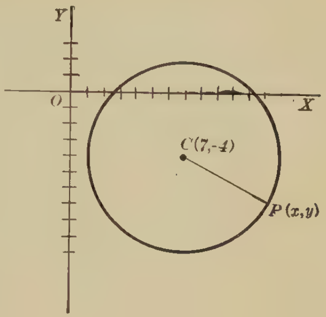
1. *Take the origin and axes in a convenient position.* The origin will usually be a fixed point mentioned in the problem, or the point midway between two such points, with one of the axes passing through them. This does not involve any loss of generality as far as the *locus* is concerned, since the locus is independent of the position of the coördinate axes; but a correct choice of axes will greatly simplify the form of the equation and the work of deriving it.

2. *Designate by $P(x, y)$ any point satisfying the given conditions and therefore on the curve.* This must not be a point in some special position, as on one of the coördinate axes, unless the problem expressly requires such a location.

3. *Draw any lines suggested by the data of the problem.* These lines may suggest auxiliary construction as well. It is sometimes helpful to draw the perpendiculars from P to the coördinate axes.

4. *Express the conditions of the problem in an equation which contains x, y, and the given constants, or which can be reduced to such a form.* For this purpose it will be necessary to find some formula or principle of geometry relating to the coördinates, the given constants, and the auxiliary lines.

5. *Simplify the equation.* It should contain all the given arbitrary constants, but *no variables other than x and y.*



Example 1. — Find the equation of the circle* of radius 6 whose center is (7, −4).

Solution. — We are to find the equation of the locus of a point whose distance from the point (7, −4) is always 6. Here the coördinate axes are fixed by the statement of the problem. Take any point $P\ (x, y)$ as one satisfying the conditions of the problem and draw CP. Then

$$CP = 6.$$

Applying the distance formula gives at once

$$\sqrt{(x-7)^2 + (y+4)^2} = 6.$$

or

$$(x-7)^2 + (y+4)^2 = 36.$$

We can either plot the curve by points, or, since we know that it is a circle, draw it with compasses.

Example 2. — A rock in the ocean lies 6 miles off a stretch of straight coast and a ship moves so as to be always equidistant from the rock and the coast. Find the locus of the ship.

Solution. — Take the coast for the x-axis and let the y-axis pass through the rock, which will have the coördinates (0, 6). Take a point $P\ (x, y)$ and draw its ordinate AP. Then $OA = x$, $AP = y$. P is to be equidistant from R and OX. This suggests drawing RP, AP being already drawn. By the conditions of the problem $RP = AP$. To get the length of RP use the distance formula, which gives

$$RP = \sqrt{x^2 + (y-6)^2}. \quad \text{But its equal } AP = y.$$

$$\therefore \sqrt{x^2 + (y-6)^2} = y, \text{ whence } x^2 - 12\,y + 36 = 0.$$

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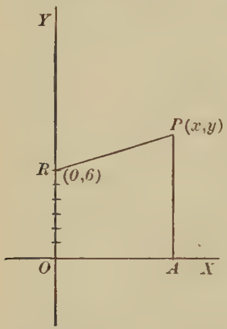
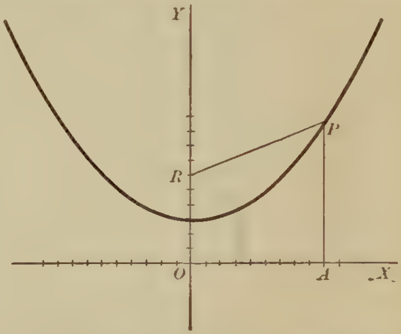
* In Analytic Geometry the word *circle* always means the same thing as *circumference.*

We have now found the *equation* of the locus. To complete the solution of the problem we plot the graph of the equation and draw the lines RP and AP to a typical point P of the graph to indicate that it actually satisfies the conditions of the problem.





EXAMPLE 3. A fixed point and a fixed line at a distance p from the point are given. Find the locus of a point which is always equidistant from the given point and the given line.

Solution. — This is obviously a generalization of Example 2. Let the fixed line be the x-axis and the fixed point be $(0, p)$. We can use the figure of Example 2 if we understand that R designates $(0, p)$, which is any point on the y-axis except the origin. Then as before, we have

$$RP = AP,$$

whence
$$\sqrt{x^2 + (y - p)^2} = y,$$

or
$$x^2 - 2py + p^2 = 0.$$

The graph of this equation is the same as that of Example 2 when $p = 6$; for other values of p we have similar curves.

The derivation of the above equations shows that the coördinates of all points which lie on the locus satisfy the equation obtained. To show that the equation fulfills the second requirement of the definition of the locus, namely, that any point the coördinates of which satisfy the equation, lies on the locus, it is merely necessary to see that the various steps taken in the derivation of the equation can be retraced. To illustrate, in Example 2 any point $P(x, y)$, whose coördinates satisfy the equation $x^2 - 12y + 36 = 0$, must also satisfy the previous equation since it can be reduced to that form by reversing the steps by which the latter was obtained. But the relation $\sqrt{x^2 + (y-6)^2} = y$ simply says that the point P is equidistant from the x-axis and R, which was the condition stated in the problem.

PROBLEMS

1. What is the equation of a straight line

(*a*) parallel to the x-axis and 6 units above it;
(*b*) parallel to the y-axis and 6 units at the left;
(*c*) parallel to the x-axis and a units distant from it;
(*d*) parallel to the y-axis and a units distant from it?

2. Solve Example 2 on page 29 when

(*a*) the origin is at the rock;
(*b*) the origin is midway between the rock and the shore.

3. Find and plot the equation of the locus of points equidistant from

(*a*) $(3, 2)$ and $(6, -2)$;
(*b*) $(-4, 3)$ and $(6, 0)$;
(*c*) $(-5, 0)$ and $(0, 4)$;
(*d*) $(2, -2)$ and $(-8, 8)$;
(*e*) $(10, 0)$ and $(-2, 4)$;
(*f*) $(0, 0)$ and $(a, 0)$.

4. Find and plot the equation of the locus of points equidistant from

(*a*) $(7, 6)$ and $(-5, 4)$;
(*b*) $(10, -6)$ and $(12, 8)$;
(*c*) $(-2, 0)$ and $(16, -8)$;
(*d*) $(9, 9)$ and $(-1, -1)$;
(*e*) $(-10, 6)$ and $(8, -10)$;
(*f*) $(a, 0)$ and $(0, a)$.

5. Find and plot the equation of the circle having
(*a*) the radius 3 and the center $(-5, 6)$;
(*b*) the radius 6 and the center $(1, -3)$;
(*c*) the radius 10 and the center $(0, 0)$;
(*d*) the radius r and the center (h, k).

6. Find and plot the equation of the circle having
(*a*) the center $(9, 0)$ and the radius 9;
(*b*) the center $(8, -6)$ and the radius 10;
(*c*) the center $(-2, 4)$ and the radius 8;
(*d*) the center $(0, r)$ and the radius r.

7. Find and plot the equation of the locus of points equidistant from
(*a*) the line $y = 6$ and the point $(3, 0)$,
(*b*) the line $x = 4$ and the point $(-2, 0)$;
(*c*) the line $x = -3$ and the point $(0, -5)$;
(*d*) the line $x = 0$ and the point $(p, 0)$.

8. Find and plot the equation of the locus of points equidistant from
(*a*) the line $x = 4$ and the point $(-2, 0)$;
(*b*) the line $x = -1$ and the point $(2, 2)$;
(*c*) the line $y = -1$ and the point $(0, 5)$;
(*d*) the line $y = p$ and the point $(0, 0)$.

9. Find the locus in Problem 7 when the distance from the point is twice that from the line.

10. Find the locus in Problem 8 when the distance from the point is one half that from the line.

11. Find the locus in Problem 7 when the distance from the line is twice that from the point.

12. Find the locus in Problem 8 when the distance from the line is one half that from the point.

13. Find the locus in Problem 8 if the distance from the point is 2 units greater than the distance from the line.

14. Find the locus in Problem 7 if the distance from the point is 2 units less than the distance from the line.

15. Find the equation of the straight line passing through

(*a*) (3, 4) and (5, −2); (*c*) (1, 1) and (−2, −2);
(*b*) (4, 3) and (−1, 6); (*d*) (a, 0) and (0, b).

16. Find the equation of the straight line passing through each pair of points in Problem 3.

17. Find the equation of the straight line passing through (−2, 3) and of inclination

(*a*) 45°; (*b*) 135°; (*c*) 150°; (*d*) 30°.

18. Find the equation of the straight line of slope 2 which passes through

(*a*) (−4, 0); (*b*) (6, 2); (*c*) (3, 5); (*d*) (8, −4).

19. The distance between two fixed points is $2c$. Find the locus of a point moving so that the sum of the squares of its distances from the points is $4c^2$.

Hint. — Let the points be $(\pm c, 0)$.

20. The base of a triangle is of length $2a$. Find the locus of the vertex if the vertical angle is 90°.

21. The ends of a straight line of variable length rest on two perpendicular lines. Find the locus of the middle point if the area of the triangle formed is a constant k.

21. Functional Variables. — The symbols used in mathematics represent two kinds of quantities, variables and constants.

A *variable* is a quantity which takes on an unlimited number of values; *e.g.* the velocity of a falling body, the abscissa of a point moving along a curve, etc., are variable quantities.

A *constant* is a quantity having a fixed value. There are two kinds of constants: *absolute* constants, which have the same value in all problems, as 2, −6, π; and *arbitrary* constants, which may have any value assigned, but keep the same value in a given discussion, as the quantities

m_1 and m_2 in the discussion of parallel and perpendicular lines in § 15.

When two variables are connected by some law such that the value of one depends upon that of the other, the first variable is said to be a function of the other. The first variable is called the *dependent* and the second the *independent* variable. Thus the area of a circle is a function of the radius, the ordinate of a point on a curve is a function of the abscissa, the pressure of steam is a function of the temperature, etc. Such relations are commonly expressed by means of equations, as, in the case of the formula expressing the area of a circle, $A = \pi r^2$.

22. Functional Notation. — The expression $f(x)$ is used to denote any function of x; it is read "function of x," or "f of x." Similarly $f(x, y)$ stands for a function of both x and y. To denote a different function of x, some other letter is used, as $F(x)$ or $g(x)$. When it is possible, functional relations are usually expressed by means of equations and in this case the expression $f(x)$ may be regarded as an abbreviation for some combination of terms containing x and no other variable. It is in this sense only that we shall use the functional notation for the present.

If an equation in x and y is solved for y, we may regard x as the independent variable and y as a function of x. Thus, if x and y are connected by the relation $x^2 + 4y^2 = 4$, $y = \pm\frac{1}{2}\sqrt{4 - x^2}$ may be denoted by the abbreviation $y = f(x)$, where $f(x)$ stands for the quantity $\pm\frac{1}{2}\sqrt{4 - x^2}$. The symbols $f(1)$, $f(-2)$ represent the values of the function when 1 and -2 respectively are substituted for x. For this function $f(1) = \pm\frac{1}{2}\sqrt{4 - 1} = \pm\frac{1}{2}\sqrt{3}$, and $f(-2) = \pm\frac{1}{2}\sqrt{4 - 4} = 0$.

If it is not desired to solve the equation for either variable, we may transpose all terms to the first member and denote this form of the equation by the expression $f(x, y) = 0$.

PROBLEMS

1. Write the equation $2x^2 + y^2 = 4xy$ in the three forms $y = f(x)$, $x = f(y)$, and $f(x, y) = 0$.

Solution. — To express y in terms of x, solve the equation for y. To do this we write the equation in the form $y^2 - 4xy + 2x^2 = 0$. This is a quadratic in y and the quadratic formula (see Introduction) gives at once:

$$y = \frac{4x \pm \sqrt{16x^2 - 8x^2}}{2} = 2x \pm x\sqrt{2}.$$

Hence $\qquad y = f(x) = 2x \pm x\sqrt{2}.$

In like manner $\qquad x = f(y) = y \pm \tfrac{1}{2}y\sqrt{2}.$

and $\qquad f(x, y) = 2x^2 - 4xy + y^2 = 0.$

2. Write each of the following equations in the three forms $y = f(x)$, $x = f(y)$, and $f(x, y) = 0$:

(*a*) $xy + y = 6$;
(*b*) $x^2 + 9y^2 = 9$;
(*c*) $x^2 + 2x = y^2 - 6y$;
(*d*) $y = (x + 1)^3$;
(*e*) $x^4 - 4x^2y + 4y^2 = 16$;
(*f*) $x^2 + x + y^2 + y = 6$;
(*g*) $x^2 - 4y^2 = 16$;
(*h*) $x^2 + y^2 = 8y$.

3. If $f(x) = x^3 - 2x^2 + 3x + 9$, find the value of $f(1)$, $f(0)$, $f(-3)$, $f(-x)$.

4. If $f(x) = 6x^4 - 5x^2 + 3$, show that $f(-x) = f(x)$ identically.

5. If $f(x, y) = 8x^2 + y - y^2$, show that $f(-x, y) = f(x, y)$ identically. Does $f(x, -y) = f(x, y)$?

6. Write a function of x and y such that

(*a*) $f(x, -y) = f(x, y)$ identically;
(*b*) $f(-x, -y) = f(x, y)$ identically;
(*c*) $f(-x, y) = f(x, y)$ identically.

23. Discussion of Curves. — Since it is possible to plot but a few points on a curve, the graph is always more or less inaccurate. Some of the properties of a curve can be found by an examination of its equation, sometimes called a *discussion*. The purpose of this discussion is threefold: it gives *exact* information regarding the curve, it furnishes a check upon the accuracy of the plot, and it usually facilitates the labor of plotting. A complete discussion of a curve in general requires the aid of the calculus. The properties that can be most conveniently studied without the calculus are the *intercepts*, *symmetry*, *extent*, and *asymptotes* of the curve.

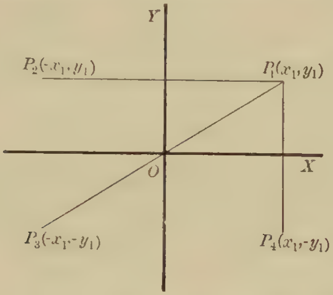
24. Intercepts. — *The intercepts of a curve are the distances from the origin to the points where it meets the axes.* In other words, the x-intercepts are the abscissas of the points where the curve meets the x-axis; the y-intercepts are the ordinates of the points where it meets the y-axis. Hence, to find the x-intercepts set $y = 0$ in the equation of the curve and solve for x; to find the y-intercepts set $x = 0$ and solve for y.

25. Symmetry. — *Definitions. The* **axis of symmetry** *of two points is the perpendicular bisector of the line joining them. The* **center of symmetry** *of two points is the point midway between them.*

A curve is symmetrical with respect to an axis or to a center when each point of the curve has its symmetrical point on the curve.

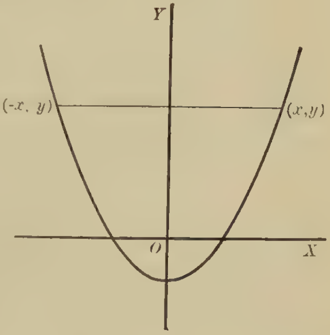
Thus the points $(3, 4)$ and $(-3, 4)$ are symmetrical with respect to the y-axis; $(3, 4)$ and $(3, -4)$ with respect to the x-axis; and $(3, 4)$ and $(-3, -4)$ with respect to the origin. From the figure we can readily establish the general principle:

Two points are symmetrical with respect to the x-axis when their abscissas are the same and their ordinates differ only in sign; to the y-axis when their ordinates are the same and their abscissas differ only in sign; to the origin when their respective coördinates differ only in sign.



The proof is left to the student.

Applying this theorem to curves, we see that a curve is symmetrical with respect to the y-axis, if for each point (x, y) on the curve, the point $(-x, y)$ is also on the curve. The symmetry of a curve may be determined by inspection of the equation according to the following theorem:



If an equation is unchanged[*] by the substitution of $-x$ for x, its locus is symmetrical with respect to the y-axis.

If an equation is unchanged by the substitution of $-y$ for y, its locus is symmetrical with respect to the x-axis.

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[*] That is, identical with the original equation or such that it is satisfied by the same values of x and y.

If an equation is unchanged by the substitution of $-x$ for x and $-y$ for y, its locus is symmetrical with respect to the origin.

The proof of the first statement is as follows:

Let the equation be written in the form

$$f(x, y) = 0. \tag{1}$$

Let $P(x_1, y_1)$ be any point on the curve and $Q(-x_1, y_1)$ be its symmetrical point with respect to the y-axis.

We must show that the coördinates of Q satisfy equation (1), *i.e.*, that

$$f(-x_1, y_1) = 0.$$

If we substitute $-x$ for x in equation (1), we obtain

$$f(-x, y) = 0. \tag{2}$$

This equation is satisfied by the same values of x and y as equation (1) by hypothesis.

Now P is on the curve, hence its coördinates satisfy equation (1) and therefore equation (2). Hence

$$f(-x_1, y_1) = 0,$$

which was to be shown.

Therefore Q is also on the curve.

Thus the definition of symmetry is satisfied and the theorem proved. The other cases are treated in exactly the same manner.

The figure at the bottom of page 37 is the graph of $x^2 - 2y - 1 = 0$, an equation satisfying the test for symmetry with respect to the y-axis. Solving for x, we have $x = \pm\sqrt{2y + 1}$, and the table of values illustrates various pairs of symmetrical points.

y	x
$-\frac{1}{2}$	0
0	± 1
1	$\pm\sqrt{3}$
2	$\pm\sqrt{5}$
4	± 3

A curve which is symmetrical with respect to both the x- and y-axes is symmetrical with respect to the origin, but the converse is not true (*e.g.*, $y = x^3$). Moreover, a curve may have axes of symmetry other than the coordinate axes. In this case, if there are two perpendicular axes of symmetry, their intersection will be a center of symmetry.

Exercise 1. Prove that if P_1 (x_1, y_1) and P_2 (x_2, y_2) are symmetrical with respect to the x-axis, then $x_1 = x_2$, and $y_1 = -y_2$.

Exercise 2. Write a proof of the third test for symmetry.

26. Extent. — To investigate the extent of a curve, we solve its equation for y in terms of x, and for x in terms of y. If either operation gives rise to radicals of even degree involving one of the variables, say x, then values of x which make the expression under the radical negative must be excluded, for the corresponding values of y would be imaginary.

Example. — Find the extent of the locus of $4x^2 - 9y^2 = 36$.

Solution. — Solving for y and x, respectively, we have

$$y = \pm\tfrac{2}{3}\sqrt{x^2 - 9}, \qquad x = \pm\tfrac{3}{2}\sqrt{y^2 + 4}.$$

For $x^2 < 9$, or for values of x between ± 3, $x^2 - 9$ is negative, and therefore y is imaginary. Hence such values of x must be excluded from the table of values and the curve lies wholly without that part of the plane bounded by the lines $x = \pm 3$.

As $y^2 + 4 > 0$ for all values of y, no value of y needs to be excluded.

PROBLEMS

1. Discuss and plot the graph of $4x^2 + y^2 - 16x = 0$.

Solution. — Setting $y = 0$, we find that the x-intercepts are 0 and 4; setting $x = 0$, we find that the only y-intercept is 0.

x	y
0	0
1	± 3.4
2	± 4
3	± 3.4
4	0

Symmetry: With respect to the x-axis.

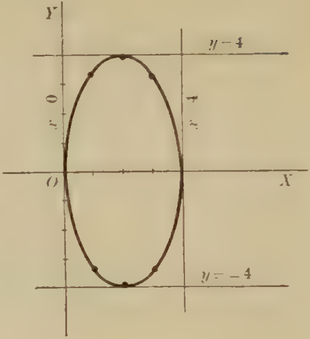
Extent: Solving for x, we have

$$x = 2 \pm \tfrac{1}{2}\sqrt{16 - y^2}.$$

Hence y^2 must be ≤ 16 and the curve lies between the lines $y = \pm 4$.

Solving for y, we have

$$\begin{aligned} y &= \pm\sqrt{16x - 4x^2} \\ &= \pm 2\sqrt{x(4 - x)}. \end{aligned}$$



By factoring the expression under the radical, we make it easy to determine the sign of this expression for various values of x. If $x > 4$ or $x < 0$, the factors under the radical have unlike signs and their product is negative. Hence such values of x must be excluded and the curve lies between the lines $x = 0$ and $x = 4$.

2. Discuss and plot the graph of $y = x^3 - a^2x$.

Solution. — This equation contains an arbitrary constant a. It will have a different graph for each value given a, but the properties obtained by the discussion will be common to all such curves. Therefore in treating any equation containing an arbitrary constant the discussion should be carried out without substituting a value for the arbitrary constant.

Intercepts: Setting $x = 0$, the y-intercept is 0. Setting $y = 0$, $x^3 - a^2x = 0$; the x-intercepts are 0, $\pm a$.

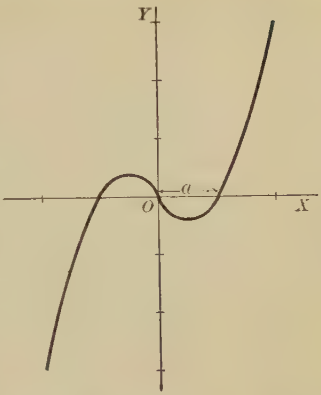
Symmetry: The substitution of $-x$ for x, or $-y$ for y, changes the equation, but if $-x$ is substituted for x and also $-y$ for y, we get

$$-y = (-x)^3 - a^2(-x),$$

or

$$-y = -x^3 + a^2x.$$

This may be reduced to the original equation by multiplying through by -1. Hence the same values of x and y satisfy both equations and the curve is symmetrical with respect to the origin.



Extent: Since this equation is of the third degree, it will have at least one real solution for any value of y. Thus we need not attempt to solve it for x. It is already solved for y and no radicals appear, hence no values of x need to be excluded, for none make y imaginary. If the equation is put in the form $y = x(x^2 - a^2)$, it becomes evident that when $x > a$, y is positive, and when x approaches infinity, y approaches infinity. Thus the curve extends to infinity in the first quadrant. Owing to the symmetry, the path where $x < 0$ is similar, extending to infinity in the third quadrant.

x	y
0	0
$\frac{1}{2}$	$-\frac{3}{8}$
1	0
2	6

We could now sketch the curve *roughly*. A table of values for x positive gives a more accurate graph. In making the table we assign to a any convenient value, as 1. etc.

Discuss and plot the graphs of each of the following equations:

3. $x^2 - 4y = 0$.

4. $y^2 - 8x + 16 = 0$.

5. $4y = 16 - x^2$.

6. $5y = x^2 - 25$.

7. $y = x^2 - 2x$.

8. $y^2 = -8x$.

9. $x^2 + y^2 - 16x = 0$.

10. $x^2 + 4y^2 + 4y = 0$.

11. $x^2 + 2y^2 = 16$.

12. $9y^2 - x^2 - 36 = 0$.

13. $x^2 - y^2 + 12y = 0$.

14. $x^2 - 4y^2 + 4x = 0$.

15. $4x^2 + 64x + y^2 = 0$.

16. $4x^2 + 64x - y^2 = 0$.

17. $y^2 = 8x^3$.

18. $y = x^3 - 9x$.

19. $x^2 = 9y^3$.

20. $y - (x - 2)^3 = 0$.

21. $y = x^4 - 4x^2$.

22. $y = x^3 + x$.

23. $4x^2 + y^2 - 16y = 0$.

24. $4x^2 + 3y^2 = 0$.

25. $y^2 = ax^3$. (Semi-cubical parabola.)

26. $y^2 = 2px$. (Parabola.)

27. $x^2 + y^2 = a^2$. (Circle.)

28. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (Ellipse.)

29. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. (Hyperbola.)

30. $x^2 = 2py$. (Parabola.)

31. $y = ax^3$. (Cubical parabola.)

32. $xy = \frac{a^2}{2}$. (Equilateral hyperbola.)

27. Horizontal and Vertical Asymptotes. — In Example 4, page 25, we found that the curve extended indefinitely along the coördinate axes. In such cases we say that each axis is an *asymptote* of the curve. In general we have the definition:

An asymptote of a curve is a straight line approached by the curve as its tracing point recedes to infinity.

Not all curves which extend indefinitely have asymptotes (Example 2, page 29), but, when asymptotes exist, they are of considerable assistance in drawing the curve. For the present we shall consider only horizontal and vertical asymptotes, *i.e.*, those parallel to the coördinate axes.

In order to develop a method for finding asymptotes let us take the equation $xy + x - 2y = 8$ as an example. If we solve first for y and then for x, we obtain

$$y = \frac{8 - x}{x - 2}$$

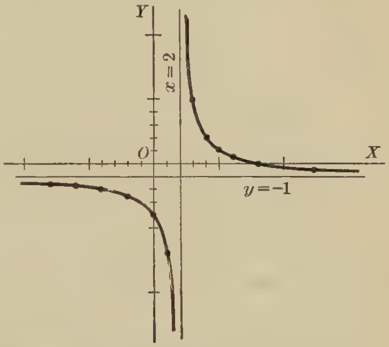
and

$$x = \frac{2y + 8}{y + 1}$$

Inspection of the first solution shows that as x approaches 2, the numerator approaches 6 and denominator 0; hence y becomes indefinitely great. Thus the line $x = 2$ is an asymptote of the curve. In like manner the second solution shows that $y = -1$ is an asymptote.

Making a table of values and using the asymptotes as guiding lines, we now draw the curve.

x	y	x	y
-8	-1.6	2	$+\infty$
-6	-1.75	3	5
-4	-2	4	2
-2	-2.5	5	1
0	-4	6	0.5
1	-7	8	0
2	$-\infty$	12	-0.4



The procedure employed in this example may be formulated in the following rule:

*To find a vertical (horizontal) asymptote, solve the equation for y (x). If the solution is a fraction, set the denominator equal to zero and solve for x (y).**

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* The horizontal asymptote can also be found from the solution for y in terms of x by finding the value approached by y as x approaches infinity. If the solution for y in terms of x contains x in both numerator and denominator, both terms of the fraction should be divided by the highest power of x. Thus, in the example just worked

$$y = \frac{8 - x}{x - 2}.$$

Dividing numerator and denominator by x, we have

$$y = \frac{\frac{8}{x} - 1}{1 - \frac{2}{x}}.$$

As x approaches infinity, $8/x$ and $2/x$ both approach 0; hence y approaches $\frac{-1}{+1} = -1$, and $y = -1$ is the horizontal asymptote.

In discussing the locus of an equation the asymptotes, if any exist, should always be found.

Example. — Discuss and plot the graph of $x^2y + 4y = 8$.

Solution. — Solving for y and x, we have respectively:

$$y = \frac{8}{x^2 + 4} \quad \text{and} \quad x = \pm 2\sqrt{\frac{2 - y}{y}}$$

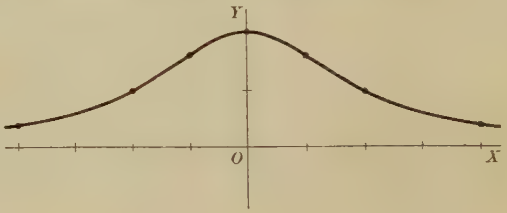
Intercepts: No x-intercept; y-intercept 2.

Symmetry: With respect to the y-axis.

Extent: The solution for y contains no radical, hence no values of x need to be excluded. The solution for x shows that x is imaginary when y is negative or greater than 2; hence we must exclude values of $y < 0$ and > 2.

Asymptotes: There is no vertical asymptote, since $x^2 + 4 = 0$ yields no real values of x. The solution for x in terms of y shows that the line $y = 0$, or the x-axis, is an asymptote.

x	y
0	2
± 1	1.6
± 2	1
± 4	0.4



PROBLEMS

Find the horizontal and vertical asymptotes and plot the graphs of the following equations.

1. $xy + 6x = 12$.
2. $xy - 4y = 16$.
3. $x^2y - 16y = 32$.
4. $(x^2 - 9)(y - 2) = 18$.
5. $(x^2 - 3x)y = 6$.
6. $x = y(x - 1)^2$.
7. $x^2y = 27 - 9y$.
8. $(x^4 - 4x^2)y = 8$.
9. $xy^2 - 16x = 16$.
10. $xy + 4x + 4y + 4 = 0$.

Discuss and plot the graphs of the following equations:

11. $x^2y = 36$.

12. $x(y^2 + 4) = 16$.

13. $xy^2 = 25$.

14. $xy = a^2$.

15. $xy = 12$.

16. $xy = -16$.

17. $(x^2 - 9)y = 12$.

18. $(x - 2)^2 y = 8$.

19. $x(y^2 - 4) = 16$.

20. $x^2y^2 = 36$.

28. Symmetrical Transformations. — If the tests for symmetry of § 25 are not satisfied, the substitutions used will transform the given equation into a new equation.

For example, if we substitute $-x$ for x in the equation of a curve symmetrical with respect to the y-axis, the equation is unchanged; but if the curve is not symmetrical with respect to the y-axis, a new equation is obtained. The question at once arises: what relation has the locus of the new equation to that of the given equation?

In this case it can be shown as in the proof of § 25, that for any point $P(x_1, y_1)$ on the locus of the given equation $f(x, y) = 0$, the symmetrical point $Q(-x_1, y_1)$ lies on the locus of the new equation $f(-x, y) = 0$. For this reason the curves are said to be symmetrical to each other with respect to the y-axis, and the transformation is called a *symmetrical transformation*. Similar reasoning applies to the other substitutions.

Summarizing, we have the following definition and theorem:

Two curves are symmetrical to each other with respect to an axis or to a center, when each point of one curve has its symmetrical point on the other.

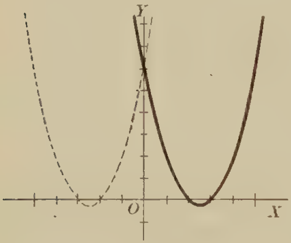
If in an equation substitution is made of

$$\begin{array}{r} -x \textit{ for } x \\ -y \textit{ for } y \\ -x \textit{ for } x \textit{ and } -y \textit{ for } y \end{array} \left\{ \begin{array}{l} \textit{the locus of the new equa-} \\ \textit{tion is symmetrical to that} \\ \textit{of the old with respect to} \\ \textit{the} \end{array} \right\} \begin{array}{l} y\textit{-axis;} \\ x\textit{-axis;} \\ \textit{origin.} \end{array}$$

EXAMPLE. — Write the equation of the curve symmetrical to $y = x^2 - 5x + 6$ with respect to the y-axis, and plot both curves on the same axes.

Solution. — Putting $-x$ for x, we get $y = x^2 + 5x + 6$. The continuous curve is the graph of the original equation, the dotted curve that of the new one.

Note that the table of values for $y = x^2 + 5x + 6$ may be obtained from that for $y = x^2 - 5x + 6$ by merely changing the signs of the values of x.



TABLES OF VALUES

$y = x^2 - 5x + 6$

x	y
0	6
1	2
2	0
3	0
4	2
5	6

$y = x^2 + 5x + 6$

x	y
0	6
-1	2
-2	0
-3	0
-4	2
-5	6

29. Transformations of Equations not Altering Loci. — The following transformations of the equation leave the form of the locus unchanged.

(*a*) Rearrangement of terms, solving for one variable, multiplying by a constant.

These do not alter the relation between x and y.

(*b*) The symmetrical substitutions.

(*c*) Interchange of variables.

The effect of this is to interchange the axes.

PROBLEMS

1. Write the equation of the curve symmetrical to the given curve with respect to the x-axis, and plot both curves on the same axes.

(*a*) $4y = x^2 - 4$; (*c*) $y = x^3 + x$;
(*b*) $x^2 + 4y^2 - 16y = 0$; (*d*) $y = x^3 - 3x^2$.

2. Write the equation of the curve symmetrical to the given curve with respect to the y-axis and plot both curves on the same axes.

(*a*) $y^2 = 4x - 4$; (*c*) $y = x^3 + x$;
(*b*) $x^2 + 4y^2 - 16x = 0$; (*d*) $y = x^3 - 3x^2$.

3. Write the equation of the curve symmetrical to the given curve with respect to the origin and plot both curves on the same axes.

(*a*) $y = x^2 - 3x + 2$; (*c*) $8y = x^3 - 12x^2$;
(*b*) $x = 2y^2 - 6y - 8$; (*d*) $y = 5x - 3$.

4. Plot on the same axes the graphs of

(*a*) $y^2 = 3x + 5$ and $x^2 = 3y + 5$;
(*b*) $y = (x - 1)^3$ and $x = (y - 1)^3$;
(*c*) $x^2 - 4y^2 = 4$ and $y^2 - 4x^2 = 4$;
(*d*) $y^2 = x^3$ and $x^2 = y^3$;
(*e*) $y = x^3$ and $y = \sqrt[3]{x}$.

30. Equations Whose Graphs cannot be Plotted. — Since real numbers alone are used as the coördinates of a point, an equation which is satisfied only by imaginary values of the variables does not define a real curve. Such equations may be most easily distinguished by transposing all terms containing variables to the left-hand member and completing the squares. If we then have the sum of several squares equal to a negative constant, the equation is obviously satisfied only by imaginary values of x and y.

PROBLEMS

1. Show that $x^2 - 2x + 2y^2 + 8y + 14 = 0$ has no real locus.

Solution. — Completing the squares in x and y, we have

$$(x - 1)^2 + 2(y + 2)^2 = -5.$$

For any real values of x and y, $(x - 1)^2$ and $(y + 2)^2$ are positive or zero; hence their sum cannot equal -5 and no real value of x and y can satisfy the equation.

2. Examine the following equations for the existence of a locus:

(*a*) $x^2 + 6x + y^2 - 4y + 17 = 0;$
(*b*) $x^2 - 3x + y^2 + 4y = -7;$
(*c*) $(x + 2)^2 + y^2 - 6y + 8 = 0;$
(*d*) $x^2 + 16 = 0;$
(*e*) $x^4 + 4y^2 + 4 = 0;$
(*f*) $x^2 + 4xy + 4y^2 + 16 = 0.$

3. For what values of k has the equation $x^2 + y^2 + k = 0$ no real locus?

4. What is the locus of

(*a*) $x^2 + y^2 = 0;$
(*b*) $(x - 3)^2 + (y + 3)^2 = 0?$

5. For what values of k will the locus be imaginary, a point, or a curve in each of the following:

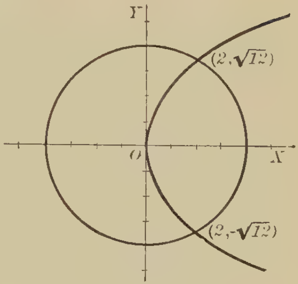
(*a*) $x^2 + 9y^2 = k^2 - 4;$
(*b*) $x^2 - 4x + y^2 + 8y + k = 0;$
(*c*) $x^2 + 4x + y^2 - 6y - k = 0;$
(*d*) $x^2 - 9y^2 + k = 0.$

31. Intersection of Curves. — It follows from the definition of the locus of an equation that a point lies on two curves if and only if its coördinates satisfy the equation of each. Hence we conclude that the coördinates of points of intersection may be obtained by solving the equations simultaneously. If there are no real solutions, the curves do not intersect.

PROBLEMS

1. Find the points of intersection of the curves $x^2 + y^2 = 16$ and $y^2 = 6x$.

Solution. — Solving simultaneously, we have $x^2 + 6x - 16 = 0$, whence $x = 2$ or -8. When $x = 2$, $y = \pm\sqrt{12}$; when $x = -8$, $y = \pm\sqrt{-48}$. Thus there are but two pairs of real solutions, giving the two points $(2, \sqrt{12})$ and $(2, -\sqrt{12})$. The figure shows the curves and their points of intersection.



Find the points common to the loci of the following equations and check the results by plotting the loci.

2. $x - y = 0$,
$y^2 + 6x = 0$.

3. $4x - y + 2 = 0$,
$x + 3y + 16 = 0$.

4. $2x + 11y - 15 = 0$,
$6x - 5y + 3 = 0$.

5. $x + y - 8 = 0$,
$x^2 - 4y + 2 = 0$.

6. $x^2 + y^2 - 25 = 0$,
$y - 2x + 11 = 0$.

7. $x^2 + y^2 - 25 = 0$,
$x^2 - 3y - 21 = 0$.

8. $x^2 + y = 7$,
$y^2 - 81 = 0$.

9. $9x^2 + 4y^2 = 36$,
$9x^2 + 16y = 27$.

10. $y^2 = 2ax$,
$x^2 = ay$.

11. $x^2 - y^2 + 36 = 0$,
$y^2 - 5x - 32 = 0$.

12. $x^2 + y^2 - 8x = 0$,
$y^2 - 8x - 2 = 0$.

13. $x^2 + 6y^2 = 20$,
$xy = 4$.

14. $3x^2 + 4y^2 = 48$,
$x^2 - y^2 = 9$.

15. $x^2 + y^2 - 6x = 0$,
$y^2 = x^3$.

16. $y = x^3$,
$y = x^2 + 2x$.

17. $xy = 4$,
$y^2 + x^2 = 10$.

18. $x + 2y = 6$,
$x^2 + 4y^2 = 20$.

19. $x^2 = 4ay$,
$$y = \frac{8a^3}{x^2 + 4a^2}.$$

a 2 1

20. $x^2 = 16y$,
$x^2 - y^2 = 64$.

21. $y = 2x$,
$x^2 + 4y^2 = 36$.

22. $y = 2x^2 - 8x$,
$2y = x^2 - 4x - 15$.

23. $y = mx$,
$x^2 + y^2 = r^2$.

24. $y = 2x - 3$,
$y^2 = 8x$.

25. $2x + y = 5$,
$x - 4y = 20$.

26. $x^2 - y^2 + 3a^2 = 0$,
$y^2 = 4ax$.

27. $x^2 - 4y^2 - 9 = 0$,
$xy - 10 = 0$.

28. Find the coördinates of the points of intersection of $y = mx$ and $y = x^3 + x$. For what values of m is there but one point of intersection?

29. Find the coördinates of the points of intersection of the straight line $y = mx$ and the hyperbola $x^2 - y^2 = 16$. For what values of m are the intersections imaginary? What happens when $m = \pm 1$?

30. Find the length of the common chord of $x^2 + y^2 = 100$ and $y^2 = 15x$.

31. Find the slope and the length of the line joining the points of intersection of $y = 4x - x^2$ and $y = x^2 - 6x$.

CHAPTER III

THE STRAIGHT LINE

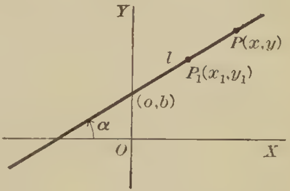
32. Equations of the Straight Line. — Several properties of the straight line, such as its determination either by two points or by a point and a direction, lead to relations between the coördinates of its tracing point that can be expressed in the form of an equation. There are several forms of the straight line equation, but all are of the first degree and for a particular line each one may in general be reduced to any of the others.

33. The Point Slope and Two-Point Forms. — Let the line l be fixed by a point and a direction. Let the point be P_1 (x_1, y_1) and let the direction of l be given by the slope $m = \tan \alpha$. Let the tracing point be $P(x, y)$. By the slope formula,

$$m = \frac{y - y_1}{x - x_1},$$

or

$$\boldsymbol{y - y_1 = m(x - x_1).} \quad (7)$$



This is the *point slope form* of the straight line equation. It is used in writing the equation of a line when one point and the slope are known. It may also be used when two points are known, for then the slope of the line can be found at once by the slope formula (3).

If in the latter case the line is determined by the points P_1 and P_2, the value of m is $\dfrac{y_1 - y_2}{x_1 - x_2}$. Substitution of this in Formula 7 gives

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1),$$

which may be written

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}. \qquad (7a)$$

This last equation is known as the *two-point form.*

34. The Slope Intercept Form. — If the slope and the y-intercept are given, the line is determined by a point on the y-axis $(0, b)$, and the slope m. Substituting in (7), we have

$$y - b = m(x - 0),$$

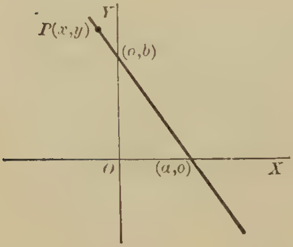
or

$$y = mx + b, \qquad (8)$$

which is called the *slope intercept form.*

This form can be used in finding the equation of a straight line when the slope and the y-intercept are known. Its most important use is in finding the slope of a line from its equation. For, if the equation of the line is reduced to form (8), by solving for y, the coefficient of x is then the slope of the line and the constant term is its y-intercept.

35. The Intercept Form. — Suppose that the intercepts of the line are given; let the x-intercept be a and the y-intercept b.



Here the line is determined by two points $(a, 0)$ and $(0, b)$. We have at once,

$$m = -\frac{b}{a}.$$

Then by (7)

$$y - b = -\frac{b}{a}(x - 0),$$

which reduces to

$$\frac{x}{a}+\frac{y}{b}=1. \tag{9}$$

This is called the *intercept form*. It is used in writing the equation of a straight line when the intercepts are known.

36. Lines Parallel to the Axes. — The equation of a line parallel to the y-axis cannot be written in forms (8) or (9), since there is no y-intercept. The equation of such a line is obviously

$$x = a.$$

Similarly the equation of a line parallel to the x-axis has the form

$$y = b.$$

Exercise 1. Derive the intercept form geometrically* when
- (*a*) a is positive and b negative;
- (*b*) a is negative and b positive;
- (*c*) both a and b are negative.

Exercise 2. Derive the slope intercept form geometrically* for the cases given in Exercise 1.

PROBLEMS

1. Derive the equation of the line determined by the points (3, 1) and (5, 4).

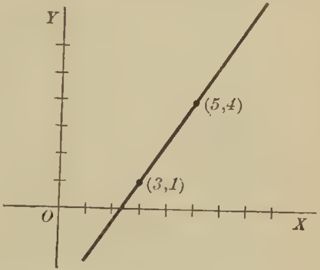
Solution. — By the slope formula $m = \frac{3}{2}$. Using the point slope form,

$$y - 1 = \tfrac{3}{2}(x - 3),$$

which reduces to

$$3x - 2y - 7 = 0.$$

The same result is obtained if the two-point form is used.



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* *I.e.*, directly from a figure, without using forms (7) or (7*a*).

Equations can be readily checked by substituting the coördinates of known points or drawing the line from the intercepts and observing whether or not it satisfies the given conditions.

2. Find the angle between the lines whose equations are $3x - 4y = 7$ and $2x + 3y = 8$.

Solution. — Solving the equations for y in terms of x, we obtain $y = \frac{3}{4}x - \frac{7}{4}$ and $y = -\frac{2}{3}x + \frac{8}{3}$. Since these are in the slope-intercept form, the slopes of the lines are $\frac{3}{4}$ and $-\frac{2}{3}$. Hence the angle between them is given by

$$\tan\beta = \frac{-\frac{2}{3} - \frac{3}{4}}{1 - \frac{1}{2}} = -\frac{17}{6},$$

and the value of the angle is 109° 26′.

3. Derive the equation of the line determined by the given points in each of the following, and check the result by finding the intercepts and drawing the line.

(*a*) $(-2, 2), (6, 5)$;
(*b*) $(4, 2), (6, -5)$;
(*c*) $(7, 5), (-4, -3)$;
(*d*) $(8, -5), (-4, 3)$;
(*e*) $(10, 10), (8, -6)$;
(*f*) $(8, -7), (-2, 6)$;
(*g*) $(6, 0), (-8, -4)$;
(*h*) $(7, -5), (1, 9)$.

4. Write the equations of the lines satisfying the following conditions:

(*a*) Passing through (3, 5) and of slope $\frac{3}{2}$;
(*b*) Passing through $(-5, 2)$ and of slope $-\frac{1}{2}$;
(*c*) Passing through $(-5, -1)$ and of slope 2;
(*d*) Passing through (0, 0) and of slope $-\frac{3}{2}$;
(*e*) Passing through (5, 1) and parallel to the y-axis;
(*f*) Passing through (1, 5) and parallel to the x-axis;
(*g*) Passing through (8, 2) and of slope 3;
(*h*) Passing through $(-2, 5)$ and of slope -2;
(*i*) Passing through (1, 5) and of slope -2;
(*j*) $a = 2, b = -6$;
(*k*) $a = -6, b = -\frac{3}{2}$;
(*l*) $a = \frac{1}{4}, b = \frac{1}{2}$;
(*m*) $a = -\frac{6}{5}, b = 2$;
(*n*) $m = \frac{1}{2}, b = -6$;
(*o*) $m = -2, b = 3$;
(*p*) $m = 4, b = -8$;
(*q*) $m = 3, b = 2$.

5. Derive the equations of the lines satisfying the following conditions:

(*a*) Passing through (8, 6) and of inclination 45°;
(*b*) Passing through (−2, 5) and of inclination 135°;
(*c*) Passing through (1, 5) and of inclination 60°;
(*d*) $m = 3, a = 4$; (*g*) $a = -4, \alpha = 135°$;
(*e*) $m = -2, a = -3$; (*h*) $b = -5, \alpha = 30°$;
(*f*) $a = -3, \alpha = 45°$; (*i*) $b = 6, \alpha = 120°$.

6. Reduce each of the following straight line equations to the slope intercept form, and draw the lines.

(*a*) $3x - 4y - 8 = 0$; (*e*) $4x - 3y = 0$;
(*b*) $\frac{x}{3} - \frac{y}{5} = 1$; (*f*) $5x + \frac{3}{2}y = 4$;
(*c*) $3x - 2y = 6$; (*g*) $2x + 5y + 3 = 0$;
(*d*) $3y + 2 = 0$; (*h*) $x + 3y = 0$.

7. Find the angle between the line* $2x - 3y = 5$ and each of the lines in Problem 6. a, b

8. Find the point of intersection of the line $2x - 3y = 5$ and each of the lines in Problem 6.

9. Two straight lines of slope −1 are tangent to a circle whose center is the origin and whose radius is 8. Find the equations of the lines.

10. Find the equation of the straight line through the point (8, 6) which has equal intercepts.

11. Find the equation of a line passing through the point (1, 3) and forming with the coördinate axes a triangle of area 8.

12. Find the equations of the two lines through the origin which trisect the triangle formed by the line $\frac{x}{16} + \frac{y}{12} = 1$ and the coördinate axes.

13. Find the area of the triangle whose sides are the lines $x = 12$, $x + 2y + 2 = 0$, and $4x - 5y = 18$.

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* More properly we should say "the line whose equation is $2x - 3y = 5$," but the close relation between a curve and its equation has given rise to the custom of using the abbreviated phrase.

14. Derive the straight line equation $x = ny + a$, where $n = \cot \alpha$, and a is the x-intercept.

15. Show that all lines for which $b = a$ have the same slope.

16. The distance between two fixed points is $2c$. Find the equation of the locus of a point moving so that its distances from the fixed points are equal.

17. The distance between two fixed points is $2c$. Find the equation of the locus of a point moving so that the difference of the squares of its distances from the fixed points is k.

18. Show analytically that the centers of circles passing through two fixed points lie on a straight line.

19. The base of a triangle is fixed in length and position. Find the locus of the opposite vertex if the slope of one side is twice the slope of the other and

(*a*) the base lies on the x-axis;
(*b*) the base lies on the y-axis.

37. The Linear Equation. — THEOREM I. *The equation of a straight line is of the first degree, in one or two variables.*

PROOF. It has been shown that the equation of a straight line cutting the y-axis may be expressed in the form $y = mx + b$. If the line is parallel to the y-axis, its equation is of the form $x = a$. Both of these are of the first degree.

THEOREM II. Conversely, *the locus of every first degree equation in one or two variables is a straight line.*

PROOF. The general form of the equation of the first degree is

$$Ax + By + C = 0, \tag{10}$$

where A, B, and C may have any values, zero included.

Case I. — When $B \neq$ * 0, the equation may be solved for y, giving

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This is of the form $y = mx + b$ and therefore its locus is a straight line having the slope $-\frac{A}{B}$ and the y-intercept $-\frac{C}{B}$.

Case II. — When $B = 0$, we have $x = -\frac{C}{A}$.

This is of the form $x = a$ and therefore its locus is a straight line parallel to the y-axis and having the x-intercept $-\frac{C}{A}$. Thus the theorem is proved for both cases.

NOTE. — For this reason an equation of the first degree is called a linear equation, the term being used by analogy for all equations of the first degree in any number of variables.

Exercise 3. Transform the general equation $Ax + By + C = 0$ into the intercept form. When is this impossible?

38. Relations between Linear Equations. — THEOREM I. *If two linear equations in two variables have the coefficients of the variables proportional, the lines defined by those equations are parallel,* and conversely.

PROOF. Let the equations be

$$Ax + By + C = 0,$$

and

$$A'x + B'y + C' = 0.$$

Solving for y,

$$y = -\frac{A}{B}x - \frac{C}{B} \quad \text{and} \quad y = -\frac{A'}{B'}x - \frac{C'}{B'}.$$

---

* This symbol has the meaning, "is unequal to."

These are in the slope intercept form.

Hence $m = -\frac{A}{B}$ and $m' = -\frac{A'}{B'}$.

But by hypothesis $A : A' = B : B'$

whence $$\frac{A}{B} = \frac{A'}{B'}$$ by alternation.

Thus we have $m = m'$, and the lines are parallel since their slopes are equal.

The converse and the special case where $B = B' = 0$ are left to the student.

Exercise 4. Assume the lines parallel and prove that the coefficients of x and y are proportional.

THEOREM II. *If in any two linear equations,*

$$Ax + By + C = 0$$

and $$A'x + B'y + C' = 0,$$

$AA' = -BB'$, *the lines defined by the equations are perpendicular*, and conversely.

PROOF. Solving the given linear equations for y, as before, we have

$$m = -\frac{A}{B} \quad \text{and} \quad m' = -\frac{A'}{B'}.$$

Then $$mm' = \left(-\frac{A}{B}\right)\left(-\frac{A'}{B'}\right) = \frac{AA'}{BB'}.$$

But by hypothesis $AA' = -BB'$, hence $mm' = -1$, and the test for perpendicularity is satisfied.

Exercise 5. Assume the lines perpendicular and prove that

$$AA' = -BB'.$$

THEOREM III. *If in any two linear equations*

$$Ax + By + C = 0$$

and $$A'x + B'y + C' = 0,$$

$A : A' = B : B' = C : C'$, *the lines defined by the equations are identical,* and conversely.

PROOF. From the assumed proportions

$$\frac{C}{B} = \frac{C'}{B'}, \qquad \text{or} \qquad b = b'.$$

Also

$$\frac{A}{B} = \frac{A'}{B'}, \qquad \text{or} \qquad m = m' \text{ as before.}$$

Hence the lines have the point $(0, b)$ in common and also have the same direction. They are therefore identical.

Exercise 6. Assume the lines identical and prove that

$$A : A' = B : B' = C : C'.$$

With the aid of these theorems it is possible to determine, by inspection of their equations, whether two lines are identical, parallel, perpendicular, or otherwise. This is of considerable aid in checking the solutions of problems involving straight lines. Suppose, for example, that it is desired to write the equation of a line passing through the point $(4, -2)$ and perpendicular to the line $10x - 4y + 11 = 0$.

From the equation the slope of the given line is $\frac{5}{2}$, and hence the slope of the required line is $-\frac{2}{5}$ (§ 15). Substituting in the point slope equation and reducing, we get $2x + 5y + 2 = 0$. This line is evidently perpendicular to the given line, for $AA' = 20$ and $BB' = -20$. Also it passes through the given point, for $2 \cdot 4 + 5(-2) + 2 = 0$.

The same problem may also be solved by using Theorem II. Any line whose equation has the form $4x + 10y = k$ will be perpendicular to the given line $10x - 4y + 11 = 0$. Since the line passes through $(4, -2)$,

$$k = 4 \cdot 4 + 10(-2) = 16 - 20 = -4.$$

Hence the required equation is

$$4x + 10y = -4, \qquad \text{or} \qquad 2x + 5y + 2 = 0.$$

39. Application to the Solution of Linear Equations. — Two equations of the first degree in two variables are classified as

(*a*) *Simultaneous* if they have one solution, which is usually the case;

(*b*) *Incompatible* if they have no common solution, as

$$\begin{cases} 3x + 7y = 1, \\ 6x + 14y = 1; \end{cases}$$

(*c*) *Dependent* if they have innumerable solutions, as

$$\begin{cases} 9x - 6y = 3, \\ 3x - 2y = 1. \end{cases}$$

It is sometimes convenient to determine the number of solutions without solving. The test by which the class of the equations can be determined is easily deduced from the relation between their loci, and from Theorems I and III, § 38.

(*a*) The equations have one and only one solution, if their lines intersect. This is the case when

$$A : A' \neq B : B'.$$

(*b*) The equations have no common solution, if their lines are parallel. This is the case when

$$A : A' = B : B'.$$

(*c*) The equations have innumerable solutions, if their lines have more than one point in common and are therefore identical. This is the case when

$$A : A' = B : B' = C : C'.$$

PROBLEMS

1. Select two pairs each of parallel and perpendicular lines from the following:

(*a*) $y = 2x + 3;$
(*b*) $2y = -6x + 9;$
(*c*) $y = 2x + 9;$
(*d*) $2y = -\frac{1}{3}x + 7;$
(*e*) $x + 5y = 0;$
(*f*) $6x + 2y + 1 = 0;$
(*g*) $5x - y - 8 = 0;$
(*h*) $x + 2y - 7 = 0.$

2. Identify the following pairs of equations as simultaneous, incompatible, or dependent:

(*a*) $2x + 6y - 7 = 0,$ $6x + 18y + 8 = 0;$
(*b*) $2x + 5y = 3,$ $2x - 5y = 8;$
(*c*) $x - 3y = 8,$ $12y - 4x = -32;$
(*d*) $2x + 2y = 15,$ $y = 2x - 5.$

3. Show that the equations $2Ax + By + D = 0$ and $Bx + 2Cy + F = 0$ are simultaneous if $B^2 - 4AC \neq 0$.

4. Find the equation of the line passing through the given point and parallel to the given line in each of the following cases:

(*a*) $(3, 0)$, $6x - 5y + 30 = 0;$
(*b*) $(-2, 6)$, $8x + 7y + 28 = 0;$
(*c*) $(4, 3)$, $3x + 4y = 12;$
(*d*) $(0, 0)$, $12x - 5y = 16;$
(*e*) $(2, 0)$, $3x + 2y = 30;$
(*f*) $(0, 0)$, $5x + 5y = 12;$
(*g*) $(3, 3)$, $3x - 4y = 15;$
(*h*) $(0, 0)$, $x + 6 = 0.$

5. In each of the following cases find the equations of two lines passing through the given point, one parallel and the other perpendicular to the given line:

(*a*) $(0, 5)$, $5x - 12y = 15;$
(*b*) $(8, 6)$, $2x + 4y = 15;$
(*c*) $(-4, 7)$, $x - 12y = 36;$
(*d*) $(6, -5)$, $6x - 8y = 27;$
(*e*) $(10, 5)$, $4x + 3y = 18;$
(*f*) $(4, 6)$, $x - y = 10;$

(*g*) (0, 0), $8x + 15y = 20$;
(*h*) (−5, 0), $y + 10 = 0$.

6. In each case in Problem 4 find the equation of a line passing through the given point and perpendicular to the given line.

7. In each case in Problem 4 find the perpendicular distance between the line and the point.

8. In each case in Problem 5, find the perpendicular distance between the line and the point.

9. In the triangle whose vertices are (0, 0), (6, 0), and (8, 6) find

(*a*) the equations of its sides;
(*b*) the equations of the perpendicular bisectors of the sides;
(*c*) the equations of the perpendiculars from the vertices on the opposite sides;
(*d*) the equations of the medians.

10. In the triangle whose vertices are (0, 0), (12, 0), and (8, 10) find

(*a*) the equations of its sides;
(*b*) the equations of the perpendicular bisectors of the sides;
(*c*) the equations of the perpendiculars from the vertices on the opposite sides;
(*d*) the equations of the medians.

11. Find the area of the triangle whose vertices are

(*a*) (4, 4), (8, −12), (12, 10); (*b*) (−5, 4), (2, −7), (9, 6).

12. Find the area of the triangle whose vertices are

(*a*) (0, 0), (8, 12), (12, −4); (*b*) (1, −1), (−7, 5), (9, 9).

13. In the triangle of Problem 10 show that

(*a*) the perpendicular bisectors meet in a point;
(*b*) the medians meet in a point;
(*c*) the perpendiculars from the vertices on the opposite sides meet in a point.

(In each case find the coördinates of the intersection of two lines and show that they satisfy the equation of the third.)

14. Show that the three points of intersection found in the above problem lie in a straight line.

15. Show that the medians of any triangle meet in a point. (Let the vertices be (0, 0) (*a*, 0), and (*b*, *c*).)

16. Show that the perpendicular bisectors of the sides of any triangle meet in a point. (Let the vertices be (0, 0), (*a*, 0), and (*b*, *c*).)

17. Show that the perpendiculars from the vertices of a triangle on the opposite sides meet in a point. (Let the vertices be (0, 0), (*a*, 0), and (*b*, *c*).)

18. Show that the three points of intersection found in Problems 15, 16, and 17 lie in a straight line.

19. Find the center and the radius of a circle circumscribed about the triangle whose vertices are

(*a*) $(0, 0), (0, -6), (4, 8)$;
(*b*) $(8, 0), (-8, 0), (12, 16)$;
(*c*) $(4, 6), (8, -14), (-12, -18)$;
(*d*) $(-10, 2), (0, 4), (6, -2)$.

20. The vertices of a parallelogram are (0, 0), (12, 0), (8, 16), and (20, 16). Lines are drawn from two opposite vertices to the mid-points of opposite sides. Show that they are parallel and that they trisect one of the diagonals.

21. Two opposite vertices of a given square are $(-4, 6)$ and $(8, -10)$. Find the other vertices and the equations of the sides.

22. Find the fourth vertex of a parallelogram which has three of its vertices:

(*a*) $(3, 5), (4, 9), (-2, 8)$; (*b*) $(6, 8), (5, -4), (0, 5)$.

How many solutions to each part?

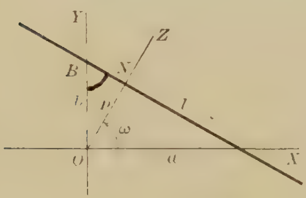
23. Find the equation of a straight line through the origin making an angle of 45° with each of the lines of Problem 1.

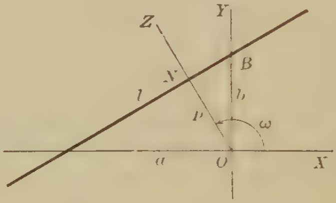
Hint. — To find the slope of the required line, observe that its inclination is 45° greater or less than that of the given line.

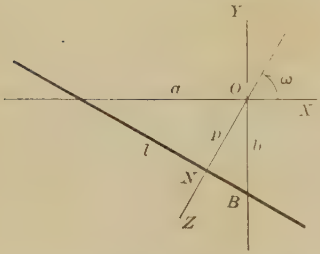
24. Show that the bisectors of the angles of any rectangle inclose a square.

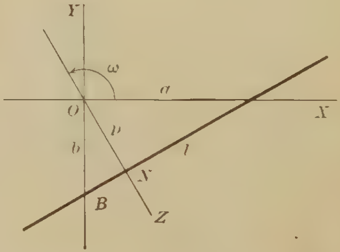
Hint. — Let the vertices be $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b).

40. The Normal Form. — The perpendicular from the origin to any line is called the *normal axis* of the line and the distance *from* the origin *to* the line measured along the normal axis is called the *normal intercept*. The inclination of the normal axis is denoted by ω and the normal intercept is denoted by p. It is obvious that the normal intercept and the y-intercept always have the same sign. The equation of the straight line determined by the constants p and ω is called the *normal form*.









Derivation of the normal form. Referring to the above figures we can easily show that in any of the four possible cases $b = \dfrac{p}{\sin \omega}$. For any case, $\sin NBO = \dfrac{p}{b}$. But NBO is equal to either ω or $180° - \omega$, since the sides of these angles are mutually perpendicular; hence $\sin NBO = \sin \omega$. This gives $\sin \omega = \frac{p}{b}$, or $b = \frac{p}{\sin \omega}$.

Since the normal axis is perpendicular to the line, its slope is the negative reciprocal of that of the line. Hence

$$m = \frac{-1}{\tan \omega} = -\frac{\cos \omega}{\sin \omega}.$$

Substituting these values in the slope intercept equation $y = mx + b$, we have

$$y = -\frac{\cos \omega}{\sin \omega} x + \frac{p}{\sin \omega}.$$

Clearing of fractions and transposing gives the normal form:

$$\boldsymbol{x \cos \omega + y \sin \omega = p.} \qquad \textbf{(11)}$$

The above proof does not cover the cases that the line is parallel to the y-axis, for there is then no slope-intercept equation. For these cases we take $\omega = 0°$ and the equation is $x = p$. This agrees with (11) since $\sin 0° = 0$ and $\cos 0° = +1$.

Exercise 7. Show that for any line the x-intercept $a = \frac{p}{\cos w}$ and derive the normal form from the intercept form.

41. Reduction of any Linear Equation to the Normal Form.

— Let the given equation of the line l be

$$Ax + By + C = 0 \qquad (1)$$

and let the normal form of the equation of l be

$$x \cos \omega + y \sin \omega - p = 0 \qquad (2)$$

Since these two equations represent the same line, we have from Theorem III, § 38, that

$$\frac{\cos \omega}{A} = \frac{\sin \omega}{B} = \frac{-p}{C}.$$

If we let r denote the common value of these ratios, we find

$$\cos\omega = rA,$$
$$\sin\omega = rB,$$
$$p = -rC.$$

Squaring the first two of these equations and adding them gives

$$1 = r^2(A^2 + B^2),$$

whence

$$r = \frac{1}{\pm\sqrt{A^2 + B^2}}.$$

Substitution of the value of r gives

$$\cos\omega = \frac{A}{\pm\sqrt{A^2 + B^2}} \qquad \sin\omega = \frac{B}{\pm\sqrt{A^2 + B^2}}$$

$$p = \frac{-C}{\pm\sqrt{A^2 + B^2}}$$

and equation (2) becomes

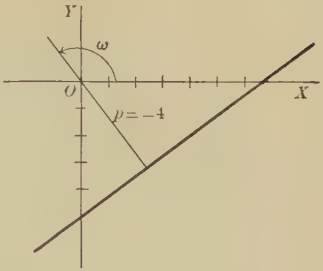
$$\frac{A}{\pm\sqrt{A^2 + B^2}}x + \frac{B}{\pm\sqrt{A^2 + B^2}}y + \frac{C}{\pm\sqrt{A^2 + B^2}} = 0.$$

This is the normal form of (1).

To determine the proper sign of the radical observe that $\sin\omega = rB$ and that $\sin\omega$ is always positive (since $\omega \leq 180°$). Hence the sign of r must be the *same* as that of B. In the special case that $B = 0$ the line is parallel to the y-axis and $\omega = 0°$. Hence $\cos\omega = 1$ and the sign of r must agree with that of A.

Summing up, we have the following working rule:

To reduce the linear equation $Ax + By + C = 0$ *to the normal form, divide it by* $\pm\sqrt{A^2 + B^2}$, *giving the radical the sign of* B.



Example. — Reduce $3x - 4y - 20 = 0$ to the normal form.

Solution. — Here $\sqrt{A^2 + B^2} = 5$, and B is negative. Therefore the required equation is

$$-\frac{3}{5}x + \frac{4}{5}y + 4 = 0.$$

This gives $\cos \omega = -\frac{3}{5}$, $\sin \omega = \frac{4}{5}$, and $p = -4$. Plotting the equation affords an excellent check on the work.

The equation $Ax + By + C = 0$ can always be reduced to the normal form. For the division by the radical is always possible, since A and B cannot both be zero. This constitutes one of the two most important advantages of the normal form. The other consists in its convenience for use in finding the distance from a line to a point.

PROBLEMS

The sines and cosines of the angles used in this group of problems should be taken from the table in the Introduction.

1. Write the equations of the lines satisfying the following data:

(*a*) $p = 6, \omega = 45°$;
(*b*) $p = 5, \omega = 150°$;
(*c*) $p = 12, \omega = 60°$;
(*d*) $p = 8, \omega = 90°$;
(*e*) $p = -16, \omega = 135°$;
(*f*) $p = -20, \omega = 150°$;
(*g*) $p = -10, \omega = 60°$;
(*h*) $p = -12, \omega = 0°$.

2. Find the equations of the lines satisfying the following data:

(*a*) $p = 10, \omega = 135°$;
(*b*) $p = 8, \omega = 120°$;
(*c*) $p = -6, \omega = 90°$;
(*d*) $p = -4, \omega = 135°$;
(*e*) $p = -12, \omega = 30°$;
(*f*) $p = -14, \omega = 30°$;
(*g*) $p = -16, \omega = 120°$;
(*h*) $p = -20, \omega = 135°$.

3. Reduce the following equations to the normal form:

(*a*) $4x - 3y = 12$;
(*b*) $8x + 15y = -12$;
(*c*) $3x + 4y = 12$;
(*d*) $12x + 5y = 0$;
(*e*) $x - y = 0$;
(*f*) $x = \sqrt{2} + y$;
(*g*) $x + \sqrt{3}y = -10$;
(*h*) $6x - 8y = 25$.

4. Reduce the following equations to the normal form:

(*a*) $3x + 4y = 24$;
(*b*) $5x + 12y = 60$;
(*c*) $8x - 15y = -60$;
(*d*) $2x + y = 20$;
(*e*) $x + 2y + 5 = 0$;
(*f*) $12x - 5y = 30$;
(*g*) $x - 4y = -16$;
(*h*) $x + y = 0$.

5. Find the value of ω for each of the lines in Problem 4 correct to the nearest minute.

6. By transformation to the normal form find the distance of a line from the origin when its intercepts are:

(*a*) 3 and 4;
(*b*) 3 and -4;
(*c*) -6 and 8;
(*d*) 6 and 9;
(*e*) 5 and -12;
(*f*) -5 and 12;
(*g*) 8 and 15;
(*h*) 7 and 12;
(*i*) 7 and $\sqrt{15}$.

7. Find the equation of the line passing through the point $(5, -10)$ and having

(*a*) $\omega = 135°$;
(*b*) $\omega = 120°$;
(*c*) $p = 2$;
(*d*) $p = -5$.

8. Prove $\tan\omega = -\cot\alpha$.

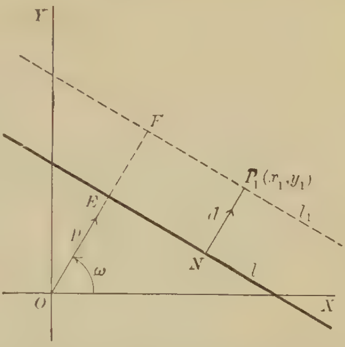
9. Find p and $\tan\omega$ in terms of a and b

10. Find $\tan\omega$ in terms of A and B.

11. Find the coördinates of the foot of the perpendicular from the origin to a line in terms of its intercepts.

12. Transform the normal form of the straight line equation into the intercept form.

42. Distance from a Line to a Point. — Let l be a line of which the equation is given and P_1 a point whose co-



ordinates (x_1, y_1) are given and let d be the distance, NP_1 from the line to the point, *reckoned from the line.*

Through P_1 draw the line l_1 parallel to l and, if necessary, produce the normal p to meet it. The normal inclination ω is the same for l and l_1, and is known from the equation of l when it is reduced to the normal form. Denote OE by p and OF by p_1. Then

$$p_1 = OF = OE + EF = p + d.$$

Hence the normal equation of l_1 is

$$p + d = x \cos \omega + y \sin \omega.$$

This is satisfied by the coördinates of P_1; hence

$$p + d = x_1 \cos \omega + y_1 \sin \omega,$$

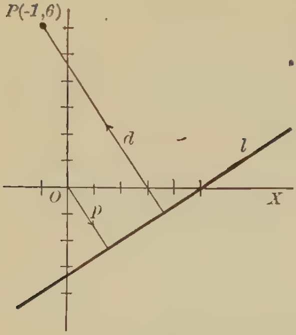
or
$$\boldsymbol{d = x_1 \cos \omega + y_1 \sin \omega - p.} \qquad \textbf{(12)}$$

But the equation of l is

$$x \cos \omega + y \sin \omega - p = 0. \qquad (11)$$

Thus the first member of (11) becomes the value of d when the coördinates of P_1 are substituted for x and y. Hence follows the rule:

To find the distance from a line to a point, reduce the equation of the line to the normal form and substitute the coördinates of the given point for the variables. The value of the first member is the required distance.



EXAMPLE. — Find the distance of the point $P(-1, 6)$ from the line $2x - 3y = 10$.

Solution. — The normal equation of l is

$$-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} + \frac{10}{\sqrt{13}} = 0.$$

$$\therefore d = \frac{(-2)(-1)}{\sqrt{13}} + \frac{3 \cdot 6}{\sqrt{13}} + \frac{10}{\sqrt{13}}$$

$$= \frac{30}{\sqrt{13}}$$

$= 8.32$, approximately.

Here p, being negative, is reckoned downward from the origin and d, being positive, is reckoned upward from the line l.

An obvious check is to draw the figure and measure the distance.

Exercise 8. Show that the distance formula can be written

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

where the radical should have the sign of B.

PROBLEMS

1. In each case find the distance from the line to the point.

(*a*) $3x - 4y = 10$, $(2, 1)$;
(*b*) $x = y$, $(0, 4)$;
(*c*) $4x - 2y + 5 = 0$, $(-3, 6)$;
(*d*) $3x + 4y = 12$, $(4, 0)$;
(*e*) $5x + 12y = 15$, $(0, 0)$;
(*f*) $x + y + 1 = 0$, $(4, 4)$.

2. In each case find the distance from the line to the point and verify the result by constructing a figure and measuring the required distance.

(*a*) $2x - y = 16$, $(8, 6)$;
(*b*) $4x - 3y = 24$, $(-10, 4)$;

(*c*) $x + 2y = 20$, $(2, -10)$;
(*d*) $3x + y = 12$, $(-4, -6)$;
(*e*) $12x + 5y = 60$, $(-4, -6)$;
(*f*) $2y - 3x = 24$, $(8, -5)$.

3. Find the distance between each of the following pairs of parallel lines by finding the distance from one line to a point on the other.

(*a*) $y = 3x + 5$,
$y = 3x - 5$;
(*b*) $3x + 2y = 12$,
$3x + 2y = 16$;
(*c*) $6x + 9y = 0$,
$4x + 6y = -24$;
(*d*) $4x - 3y = 24$,
$4x - 3y = 0$.

4. Find the distance from the intersection of the lines

(*a*) $x - 4y - 12 = 0$
$3x - 4y - 12 = 0$ to the line $x + y + 10 = 0$;
(*b*) $3x - y - 2 = 0$
$5x + y - 12 = 0$ to the line $12x - 5y + 23 = 0$;
(*c*) $7x - 5y = -6$
$4x - 11y = 21$ to the line $x + 2y = 10$.

5. Find the areas of the triangles formed by the sets of lines in Problem 4.

6. Find the areas of the triangles whose vertices are

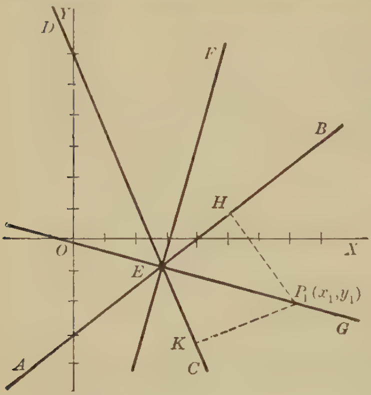
(*a*) $(0, 0)$, $(-1, 2)$, $(5, 7)$;
(*b*) $(-4, 3)$, $(2, -4)$, $(0, 6)$;
(*c*) $(-7, 8)$, $(-7, -6)$, $(1, -4)$;
(*d*) $(5, 4)$, $(8, 3)$, $(6, 5)$;
(*e*) $(-2, 10)$, $(5, 3)$, $(3, -4)$;
(*f*) $(-7, 4)$, $(4, 6)$, $(1, -4)$.

7. Find the areas of the polygons having the following vertices:

(*a*) $(6, 0)$, $(7, 10)$, $(-3, 5)$, $(-2, -6)$;
(*b*) $(2, 0)$, $(1, 4)$, $(-2, 2)$, $(-1, -3)$, $(1, -2)$;
(*c*) $(4, 3)$, $(-2, 5)$, $(-3, -4)$, $(6, -2)$;
(*d*) $(0, 0)$, $(12, 0)$, $(10, -4)$, $(4, -8)$.

8. Find the equations of the bisectors of the angles between the lines $3x - 4y = 12$ and $12x + 5y = 30$.

Solution. — Let the lines meet at E; there are two bisectors EF and EG. Let $P_1(x_1, y_1)$ lie upon EG, the bisector of the angle BEC. Then by plane geometry the perpendiculars HP_1 and KP_1 are of equal length.



To find the length of HP_1 use Formula 12.

Then

$$HP_1 = \frac{-3x_1 + 4y_1 + 12}{5}.$$

Similarly

$$KP_1 = \frac{12x_1 + 5y_1 - 30}{13}.$$

But HP_1 is negative and KP_1 positive.

Therefore

$$HP_1 = -KP_1, \text{ or}$$

$$\frac{-3x_1 + 4y_1 + 12}{5} = \frac{-12x_1 - 5y_1 + 30}{13}.$$

Simplifying and dropping subscripts, the equation of EG is

$$21x + 77y + 6 = 0.$$

In like manner the equation of EF is

$$99x - 27y - 306 = 0.$$

A convenient check on the accuracy of the work is the application of the test for perpendicularity to the two solutions.

9. Find the equations of the bisectors of the angles between

(*a*) $3x + 4y = 5,$ $\quad$ (*b*) $3x - 4y = 7,$

$\quad\;\;$ $4x + 3y = 6;$ $\quad\quad$ $4x - 3y = 8.$

10. Find the locus of points twice as far numerically from the first line as from the second in each of the groups in Problem 3.

11. Find the locus of points the sum of whose distances from each of the pairs of lines in Problem 9 is 12.

12. Find the locus of points 6 units distant from the line

(*a*) $3x + 4y = 5$; (*c*) $15x - 8y = 12$;
(*b*) $12x - 5y = 17$; (*d*) $4x - 3y = 24$.

13. A triangle has vertices (3, 0) and (−1, 3). Find the locus of the third vertex if the area is to be 16.

14. Find the equation of the line halfway between the parallels

(*a*) $2x + y = 7$, $2x + y = 10$; (*b*) $3x - 2y + 9 = 0$, $3x - 2y = 16$.

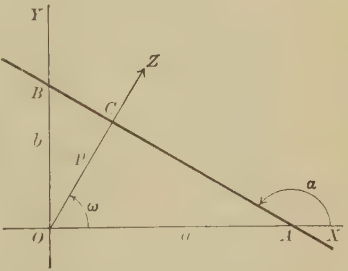
15. A triangle has the vertices (0, 0), $(12\frac{1}{2}, 0)$, and (8, 6).

(*a*) Find the equations of the bisectors of the angles.

(*b*) Show that these bisectors meet in a point.

(*c*) Find the coördinates of the center and the radius of the inscribed circle.

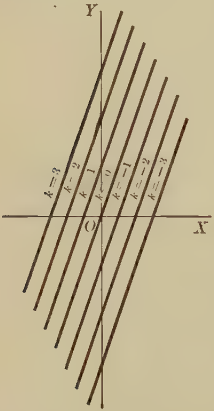
43. Geometric Conditions Determining a Line. — In each of the standard equations $y = mx + b$, $\frac{x}{a} + \frac{y}{b} = 1$, and $x \cos \omega + y \sin \omega - p = 0$, we note that there are involved two arbitrary constants which represent geometric conditions determining the line. Three of these, *a*, *b*, and *p*, represent distances, while the others, *m* and *ω*, determine the direction of the line. The direction may also be given by the inclination *α*.



Thus we associate with any straight line six constants; three, *a*, *b*, and *p*, representing distances, and three, *m*, *α*, and *ω*, representing directions. Any pair of them (one being a distance) will determine the line (and also the other constants), and combined with its current coordinates they will furnish an equation of the line. Of the many possible equations involving these constants, only the three given above are commonly used.

44. Systems of Straight Lines. — When one of the constants of a straight line equation is arbitrary and the other absolute, as in $y = 3x + k$, the equation defines a line for each value assigned to the arbitrary constant, but each line so defined has the property given by the numerical constant.

Thus in the above equation k is the y-intercept, and it may have any value; but the slope of each line is 3, and by assigning all possible values to k we have all possible lines of slope 3. Such a group of lines is called a *system*, and the arbitrary constant is called the *parameter* of the system. In the particular case under discussion we have a system of parallel lines of slope 3, of which several are drawn in the figure.



We have already seen that any of the standard forms of the straight line equation involves two arbitrary constants which correspond to the geometric conditions defining the line. Hence if one of these constants is given a numerical value and the other left arbitrary, we get a *system of straight lines* characterized by the geometrical property defined by the numerical constant, and with one geometrical property left arbitrary or variable. This correspondence is general, as will be seen later in the case of the circle and the conics.

PROBLEMS

1. Draw the lines satisfying the following conditions:

(*a*) $a = -5, \alpha = 30°$;
(*b*) $b = 8, \alpha = 60°$;
(*c*) $p = 6, \alpha = 120°$;
(*d*) $a = 10, \omega = 150°$;
(*e*) $b = -8, \omega = 120°$;
(*f*) $p = -3, \omega = 45°$;
(*g*) $a = -10, p = 6$;
(*h*) $b = -10, p = -8$
(*i*) $m = -2, p = 12$;
(*j*) $a = 12, m = 2$.

2. With *ruler and compasses only* construct the required line, given

(*a*) a and α;
(*b*) b and α;
(*c*) p and α;
(*d*) a and ω;
(*e*) b and ω;
(*f*) p and ω;
(*g*) a and p;
(*h*) b and p;
(*i*) m and p.

3. In each case in Problem 1 find the other four geometric constants.

4. Find the equation of each line in Problem 1.

5. Write the equations of the systems of lines defined as follows and draw several lines of each system.

(*a*) passing through $(5, -1)$;
(*b*) passing through $(-2, 2)$;
(*c*) passing through $(6, -3)$;
(*d*) with the y-intercept -2;
(*e*) with the x-intercept -3;
(*f*) having the slope –
(*g*) having the slope ;
(*h*) distance from origin 13;
(*i*) distance from origin 5;
(*j*) having equal intercepts.

6. What geometric conditions define the systems of lines represented by the following equations:

Hint. — Put each equation in a standard form.

(*a*) $x + k = 0$;
(*b*) $3x + 4y + k = 0$;
(*c*) $x + 3ky - 5 = 0$;
(*d*) $kx + 2y - 6 = 0$;
(*e*) $2x - ky + 4k = 0$;
(*f*) $kx + ky - 6 = 0$;
(*g*) $kx - y\sqrt{1 - k^2} = 6$;
(*h*) $y - 1 = mx + 2m$?

7. Determine k so that

(*a*) the line $3x + 4y = k$ passes through $(-2, 1)$;
(*b*) the line $5x - 4y = k$ passes through $(-3, 4)$;
(*c*) the line $kx - 2y = -3$ has the slope 2;
(*d*) the line $3x - 5y - k = 0$ has its x-intercept $= 3$;

(*e*) the line $7x - 4ky + 5 = 0$ has its y-intercept $= -3$;

(*f*) the line $4x - 3y + k = 0$ is distant 5 units from the origin.

8. Write the equation of the system of lines perpendicular to each line in Problem 7.

9. Find the equation of the system of lines whose normal axis has the slope $-\frac{5}{12}$.

10. Find the equation of the system of lines whose normal axis has the inclination 135°.

11. Write the equation of the system of lines passing through the point (8, 5). Find the value of the parameter for which the line forms with the coördinate axes a triangle of area 80.

12. In Problem 6 find the values of the parameter for which the line is 2 units distant from the origin.

13. In Problem 6 find the value of the parameter for which the y-intercept is -12.

45. Systems of Lines through the Intersection of Two Given Lines.

— Suppose that it is desired to find the equation of the system of lines through the intersection of two given lines. One way of doing this would be to solve the two given equations for the point of intersection and to use the point slope equation. A method more expeditious and of great theoretical importance follows.

THEOREM. *Let* $l_1 = 0$ and $l_2 = 0$ *be the equations** *of two intersecting lines* l_1 *and* l_2, *and let* k *be an arbitrary constant. Then* $l_1 + k \cdot l_2 = 0$ *defines a system of lines passing through the intersection of the given lines.*

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* The equation $l_1 = 0$ stands for an expression of the form

$$A_1x + B_1y + C_1 = 0,$$

and would more properly be written $l_1(x, y) = 0$, but the variables are omitted for convenience.

PROOF. — We first observe that $l_1 + k \cdot l_2 = 0$ defines a system of straight lines with k as a parameter, since it is obtained by adding two first degree equations.

Let the point of intersection P have coördinates (a, b). By the locus definition the equations $l_1 = 0$ and $l_2 = 0$ are both satisfied by $x = a$ and $y = b$, since $P\,(a, b)$ lies on both curves. Hence, the quantities $l_1\,(a, b)$ and $l_2\,(a, b)$ are both zero. Therefore $l_1\,(a, b) + k \cdot l_2\,(a, b) = 0$ for all values of k.

Hence every line of the system $l_1 + k \cdot l_2 = 0$ passes through P, since the coördinates of P satisfy the equation.*

EXAMPLE. — Find the line containing the point (2, 3) and the intersection of $l_1 : 3\,x + y - 5 = 0$ and $l_2 : x + 2\,y - 3 = 0$.

Solution. — The system $l_1 + k \cdot l_2 = 0$ contains the intersection, and k may be found for the assumed point by substituting

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* We can also prove that *every* line passing through P (with the exception of l_2) belongs to the system. Let $l_3 = 0$ be another line through P, and let $P_1\,(x_1, y_1)$ be some other point on l_3. If the coördinates of P_1 be substituted in $l_1 + k \cdot l_2 = 0$, we obtain an equation

$$l_1\,(x_1, y_1) + k \cdot l_2\,(x_1, y_1) = 0$$

in which k, being the only unknown, can be found. For this value of k, the locus of the equation $l_1 + k \cdot l_2 = 0$ passes through P_1, since its coördinates make the left hand member equal to zero. As it already passes through P, it must be identical with l_3, having two points in common with it.

When $k = 0$ the equation of the system becomes $l_1 = 0$, but for no value of k does it reduce to $l_2 = 0$. If we divide the equation through by k, we obtain $\frac{1}{k} l_1 + l_2 = 0$. This form of the equation shows that as k approaches infinity the corresponding lines of the system approach the line whose equation is $l_2 = 0$. In like manner the equation $y - 3 = m\,(x - 2)$ represents all lines passing through (2, 3) except the one parallel to the y-axis. This one has an infinite slope and its equation is $x - 2 = 0$.

the coördinates of that point in the equation of the system, thus:

$$3x + y - 5 + k(x + 2y - 3) = 0$$

becomes

$$6 + 3 - 5 + k(2 + 6 - 3) = 0,$$

and $k = -\frac{4}{5}$; from which we have the result $11x - 3y - 13 = 0$.

Verification. Solving the given equations simultaneously, the intersection is $(\frac{7}{5}, \frac{4}{5})$. These coördinates and the coördinates (2, 3) both satisfy the new equation.

The above proof will apply to any two equations of the form $f(x, y) = 0$, as well as to two linear equations. The general principle may be stated as follows:

The equation $f(x, y) + k \cdot g(x, y) = 0$ defines a system of curves passing through the intersections of the curves defined by the two equations $f(x, y) = 0$ and $g(x, y) = 0$.

PROBLEMS

1. Find by the method of the example given above the equation of the system of lines passing through the intersection of $3x - 2y = 16$ and $3x + 6y = 1$. Find the particular line of the system passing through $(-5, 4)$.

2. Find the line of the above system which has the slope $\frac{1}{2}$.

Hint. — The equation in k is

$$3x - 2y - 16 + k(3x + 6y - 1) = 0,$$

or

$$(3k + 3)x + (6k - 2)y - k - 16 = 0.$$

Solve this equation for y and the coefficient of x is the slope.

3. Find the line of the above system which is parallel to the x-axis; also the line which is parallel to the y-axis.

4. Find by the method of the illustrative example the equations of the lines determined by

(*a*) $P(4, 3)$ and the intersection of $4x - 2y = 7$ and $x - 2y = 3$;

(*b*) $P(3, 0)$ and the intersection of $3x - 2y = -12$ and $3x + 2y = 0$;

(*c*) $P(-5, 0)$ and the intersection of $3x + 2y = 5$ and $x = 6$;

(*d*) $P(-4, -1)$ and the intersection of $x - y + 5 = 0$ and $x + y - 5 = 0$.

5. Find by the method of Problem 2 two lines of the system containing the intersection of the first pair of lines, one parallel, the other perpendicular to the third line:

(*a*) $3x - 5y + 5 = 0$, $5x + 4y = 10$, and $x + y = 6$;

(*b*) $x + y = 0$, $x - y = 0$, and $2x + 3y + 5 = 0$;

(*c*) $6x - 5y + 4 = 0$, $x + y = 1$, and $x = 3$;

(*d*) $6x - 3y + 9 = 0$, $4x + 3y = 10$, and $5x - 9y + 10 = 0$.

6. Solve Problem 4 by finding the point of intersection and using Formula 7.

7. Find and plot the equations of two curves which pass through the intersections of

(*a*) $y^2 = 8x + 16$,
$y^2 = 4x + 16$;

(*b*) $x^2 + y^2 = 13$,
$x^2 - y^2 = 5$.

8. In the system of lines passing through the intersection of each pair of lines in Problem 4, plot the lines for $k = \pm\frac{1}{2}$, ± 1, ± 2, ± 3, and ± 4. What happens as k approaches 0 and $+\infty$?

46. Plotting by Factoring. — THEOREM. *If* $l_1 = 0$ *and* $l_2 = 0$ *define the lines* l_1 *and* l_2, *the locus of* $l_1 \cdot l_2 = 0$ *consists of these lines.*

PROOF. This follows directly from the locus definition. For the coördinates of a point (a, b) on either line, say l_1, satisfy its equation $l_1 = 0$. But if $l_1(x, y)$ vanishes for $x = a$ and $y = b$, the product of $l_1(x, y)$ and $l_2(x, y)$ must vanish for the same values, *i.e.*, (a, b) lies on the locus of $l_1 \cdot l_2 = 0$.

Conversely, if (a, b) is on this locus, one of the factors $l_1(x, y)$, $l_2(x, y)$ must vanish for $x = a$, $y = b$. Hence (a, b) lies on the line defined by the equation formed by setting this factor equal to zero.

The utility of this theorem is made clear by the following illustration.

Consider the equation

$$x^2 - 4y^2 - x - 2y = 0.$$

Factoring, we have

$$(x + 2y)(x - 2y - 1) = 0.$$

Now the coördinates of any point on the line $x + 2y = 0$, as $(2, -1)$, make the first factor zero and hence the product is zero. Therefore all points on $x + 2y = 0$ lie on the locus of the given equation. The same is true of $x - 2y - 1 = 0$. Conversely, if a point lies on the locus of $x^2 - 4y^2 - x - 2y = 0$, its coördinates satisfy the equation and must make at least one of the factors of the left-hand member zero. Therefore all points of the given locus belong to one of the lines $x + 2y = 0$, $x - 2y - 1 = 0$.

This reasoning applies to all equations of the form $f(x, y) = 0$, whether they define straight lines or not, although it is useful chiefly for linear factors. It leads to the following rule.

Rule for Plotting by Factoring. — *Transpose all terms to the first member. Factor as far as possible; set each factor equal to zero, and plot the resulting equations on the same axes.*

PROBLEMS

1. Plot by factoring the pairs of lines defined by the following equations:

(*a*) $x^2 - 9y^2 = 0$;
(*b*) $x^2 + xy - 2y^2 + 3x - 3y = 0$;
(*c*) $x^3 - 3x^2y - 4xy^2 = 0$;
(*d*) $x^2 - x - y^2 + y = 0$;
(*e*) $4x^2 + 4xy + y^2 + 2x + y - 2 = 0$;
(*f*) $x^2 + 2x + 1 - y^2 = 0$;

(*g*) $y^2 + 3y + 2 = 0$;
(*h*) $x^2 - 4y^2 - x - 2y = 0$.

2. Draw the locus defined by:
(*a*) $5xy^2 = 7x^2y$; (*b*) $x^3y - 4xy^3 = 0$.

3. Find the area of the triangle defined by:
(*a*) $x^2y - 2xy^2 + 3xy = 0$;
(*b*) $(x + 3)(xy - y^2) = 0$;
(*c*) $x^3 - 6x^2 - xy^2 + 6y^2 = 0$.

4. The sides of a quadrilateral are defined by $y^2 - 4xy + 4x^2 - 3y + 6x = 0$ and $x^2 + 4x + 3 = 0$. Prove that it is a parallelogram.

5. Plot the locus of the equation:
(*a*) $(x^2 + y^2 - 25)(x^2 - y^2 - 25) = 0$;
(*b*) $x^3y + xy^3 = 16xy$.

6. Show that $Ax^2 + Bx + C = 0$ defines a pair of lines parallel to the y-axis if $B^2 - 4AC > 0$, a single line if $B^2 - 4AC = 0$, and an imaginary locus if $B^2 - 4AC < 0$.

7. Write a single equation defining the locus of each of the following pairs of lines.
(*a*) $x - y = 4, x + y = 4$;
(*b*) $2x - y = 6, 2x + y = 4$;
(*c*) $2y = x + 3, 2y = x - 7$;
(*d*) $3x + 4y + 12 = 0, 3x - 4y - 12 = 0$.

8. Show that the locus of $Ax^2 + Bxy + Cy^2 = 0$ is
(*a*) a pair of intersecting lines if $B^2 - 4AC > 0$;
(*b*) a single line if $B^2 - 4AC = 0$;
(*c*) imaginary if $B^2 - 4AC < 0$.

9. Find the equations of the bisectors of the angles between the lines defined by $x^2 - 4y^2 + 3x + 6y = 0$.

10. The sides of a parallelogram are $y^2 - 6y = 0$ and $(y - 2x)^2 + 6(y - 2x) = 0$. Show that the perpendiculars upon a diagonal from opposite vertices are of equal length.

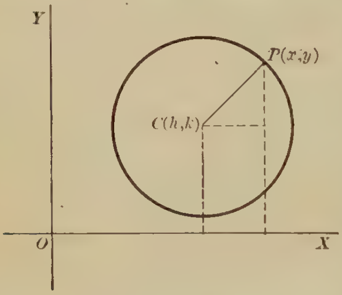
11. The vertices of a triangle are (0, 0), (5, 0), (0, −3). Write an equation defining its three sides.

12. Generalize and solve Problem 10.

CHAPTER IV

THE CIRCLE

47. The Standard Equation of the Circle. — While the equation of the straight line is expressed in several standard forms, depending on the choice of the geometric constants determining it, we use only one such form for the equation of a circle in rectangular coördinates. The constants used in determining the circle are the coördinates of its center and the length of its radius.



Let the coördinates of the center of the circle be (h, k), the length of the radius r, and the coördinates of the point tracing the circle (x, y). By the distance formula,

$$r = \sqrt{(x-h)^2 + (y-k)^2}.$$

This gives

$$(x-h)^2 + (y-k)^2 = r^2 \tag{13}$$

as the standard form of the equation of the circle.

When the center is at the origin, form (13) becomes

$$x^2 + y^2 = r^2. \tag{13a}$$

PROBLEMS

1. In each of the following cases write the equation of the circle in the standard form and simplify the result; also check the work by finding the intercepts and drawing the circle:

(*a*) center at (3, 5) and radius 5;
(*b*) center at $(-8, -6)$ and radius 10;
(*c*) center at $(4, -3)$ and radius 5;
(*d*) center at $(-8, 6)$ and radius 8;
(*e*) center at (6, 0) and radius 6;
(*f*) center at (1, 1), radius 12;
(*g*) center at (4, 5), radius 10;
(*h*) center at $(-6, 6)$, radius 12;
(*i*) center at $(3, -4)$, radius 8;
(*j*) center at $(-5, 12)$, radius 13.

2. Write the forms which the equation of the circle takes in the following cases:

(*a*) center on the x-axis;
(*b*) center on the y-axis;
(*c*) circle touching the x-axis;
(*d*) circle touching the y-axis;
(*e*) circle touching both axes.

3. Write the equations of the circles satisfying the following conditions and draw the figure in each case:

(*a*) tangent to the x-axis, radius 6, abscissa of center 4;
(*b*) tangent to the y-axis, radius 8, ordinate of center -6;
(*c*) tangent to both axes, radius 16, center in the third quadrant;
(*d*) passing through the origin, radius 10, abscissa of center -8;
(*e*) passing through the origin, radius 13, abscissa of center -12;
(*f*) passing through (2, 2), radius 5, ordinate of center -1.

4. Find the equation of the circle whose center is $(8, -6)$ and which passes through (6, 8).

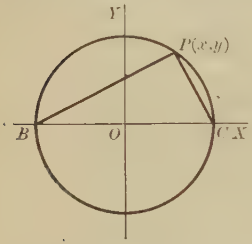
5. Find the equation of the circle which has the line joining (6, 2) and $(-8, 6)$ as a diameter.

6. Find the equation of the circle which has the center $(r, 0)$ and is tangent to the y-axis.

7. Find the equation of the circle whose radius is 8 and whose center is at the intersection of the lines $2x + 5y = 10$ and $2x - 5y = 14$.

8. Prove that an angle inscribed in a semicircle is a right angle.

Solution. — Let the radius of the circle be r; take the diameter as the x-axis and the center as the origin. Let $P(x, y)$ be any point on the circumference. We have to prove that BPC is a right angle.



Now the coördinates of B and C are $(-r, 0)$ and $(r, 0)$. Hence by the slope formula, the slope of BP is $\frac{y}{x+r}$; that of CP is $\frac{y}{x-r}$. Calling these slopes m and m', we have

$$mm' = \frac{y^2}{x^2 - r^2}.$$

But $P(x, y)$ is on the circle. Therefore, $x^2 + y^2 = r^2$, or $y^2 = r^2 - x^2$. This gives

$$mm' = \frac{r^2 - x^2}{x^2 - r^2} = -1,$$

which proves that BP and CP are perpendicular, or that BPC is a right angle.

9. Prove without the use of slopes that an angle inscribed in a semicircle is a right angle.

10. Prove that a perpendicular from any point on the circumference of a circle to a fixed diameter is a mean proportional between the segments into which it divides the diameter.

11. Prove that angles inscribed in the same segment of a circle are equal.

Hint. — Let the ends of the segment be (c, d) and $(-c, d)$ and use Formula 6.

12. Prove that a line from the center of a circle bisecting a chord is perpendicular to it.

Hint. — Let the ends of the chord be $(-r, 0)$ and (b, c).

13. A regular octagon is inscribed in a circle of radius r. Show that the length of a side is $r\sqrt{2-\sqrt{2}}$.

48. The General Equation of the Circle.

— If we expand the standard form (13), we get

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is of the form

$$\boldsymbol{x^2 + y^2 + Dx + Ey + F = 0,} \qquad \textbf{(14)}$$

which may be called the *general form* of the circle equation. Conversely, this form represents a circle, since by completing squares it is possible to reduce it back to form (13), which we know is the equation of a circle of center (h, k) and radius r. (Note exception, § 49.)

The general equation of the second degree has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Comparison with (14) shows that this can be reduced to the same form if $B = 0$ and $A = C$, by dividing by A. Hence we have the theorem:

An equation of the second degree in two variables defines a circle when, and only when, the coefficients of x^2 and y^2 are equal and the term in xy is missing.

49. Identification of Center and Radius.

— To find the center and the radius of a circle when the equation is given, reduce the equation to form (13).

Example. — Find the center and the radius of the circle

$$2x^2 - 8x + 2y^2 + 6y - 21 = 0.$$

Solution. — Dividing by 2 and completing the squares, we have

$$x^2 - 4x + 4 + y^2 + 3y + \tfrac{9}{4} = \tfrac{21}{2} + 4 + \tfrac{9}{4},$$

or

$$(x - 2)^2 + (y + \tfrac{3}{2})^2 = \tfrac{67}{4}.$$

This is of form (13) and therefore represents a circle whose center is $(2, -\frac{3}{2})$ and whose radius is $\frac{1}{2}\sqrt{67} = 4.1$ approximately.

Degenerate Forms. — Sometimes on completing the squares the term corresponding to r^2 is negative, or zero. In the former case, it may be shown by the methods of § 30 that the locus is imaginary and we say that the graph is an *imaginary circle.* In the latter case, the radius of the circle is zero, and the graph is called a *point circle.* Such forms are sometimes known as degenerate forms.

50. Special Forms. — Comparing the general form (14) and the expansion of the standard form (13) in § 48, we see that

$$D = -2h, \quad E = -2k, \quad \text{and} \quad F = h^2 + k^2 - r^2.$$

The first two of these relations enable us to determine the center by inspection. They also enable us to determine the following special forms:

if $D = 0$, $h = 0$ and the center is on the y-axis;

if $E = 0$, $k = 0$ and the center is on the x-axis;

if $D = 0$ and $E = 0$, the center is at the origin;

if $F = 0$, $h^2 + k^2 = r^2$, and the origin is on the circumference.

Exercise 1. Prove by considerations of symmetry that if $D = 0$, the center is on the y-axis.

PROBLEMS

1. Find the center and the radius of each of the following circles. If the circle is real, illustrate by a figure.

(*a*) $(x + 3)^2 + (y - 4)^2 = 64$;

(*b*) $(x + 4)^2 + (y - 5)^2 = 0$;

(c) $x^2 + y^2 - 4x + 6y - 23 = 0$;
(d) $x^2 + y^2 + 16y = 0$;
(e) $3x^2 + 3y^2 = x$;
(f) $x^2 + y^2 + 2x + 8y + 1 = 0$;
(g) $2x^2 + 2y^2 - 16x - 20y = 0$;
(h) $x^2 + y^2 + 5x - 3y - \frac{1}{2} = 0$;
(i) $x^2 + y^2 - 24x + 10y = 0$;
(j) $x^2 + y^2 + 2x + 2y + 2 = 0$.

2. Find the center and the radius of each of the following circles. If the circle is real, illustrate by a figure.

(a) $(x - 6)^2 + (y + 4)^2 = 49$;
(b) $x^2 + y^2 + 10x + 16y = 11$;
(c) $x^2 + y^2 + 14x - 7y = 28$;
(d) $2x^2 + 2y^2 - 22x - 22y = 7$;
(e) $x^2 + y^2 - 18x + 18y + 81 = 0$;
(f) $x^2 + y^2 + 16x + 30y = 0$;
(g) $x^2 + y^2 + 10x - 8y + 41 = 0$;
(h) $x^2 + y^2 - 20y + 125 = 0$;
(i) $x^2 + y^2 - 16x + 16y = 0$;
(j) $x^2 + y^2 + 8x - 15y = 0$.

3. Determine by inspection which of the circles in Problem 1

(a) have their centers on the x-axis;
(b) have their centers on the y-axis;
(c) pass through the origin.

4. Determine by inspection the centers of the circles in Problem 2.

5. Write the equations of the circles symmetrical to those of Problem 1 with respect to the x-axis and in each case draw both circles.

6. The equation of a certain system of circles is $(x + 4)^2 + (y - 3)^2 = k$. What is the common center of the circles? For what values of k are the circles real? For which of the following values of k are the x-intercepts real:

$$k = 0, 1, 4, 9, 16, 25, 36.$$

7. For what values of k do the following equations have a locus?

(*a*) $x^2 + y^2 + ky = -8$; (*b*) $x^2 + y^2 + kx = -6$.

8. Find the values of k for which each of the equations of Problem 7 defines a point.

9. Find the points of intersection and the equation of the common chord of the circles:

(*a*) $x^2 + y^2 + 8y = 20,$ (*b*) $x^2 + y^2 + 4x - 60 = 0,$
$x^2 + y^2 - 8x = 4$; $x^2 + y^2 - 4y - 36 = 0.$

10. Find the equation of the locus of the vertex of a right triangle which has the ends of its hypotenuse at

(*a*) $(4, 0)$ and $(-2, 6)$; (*b*) $(-4, 3)$ and $(0, 5)$.

11. Find the equation of the tangent to each of the following circles at the point indicated.

Hint. — The slope of the tangent is the negative reciprocal of the slope of the radius drawn to the point of tangency.

(*a*) $x^2 + y^2 + 16y - 105 = 0, (5, 4)$;
(*b*) $x^2 + y^2 - 8x - 84 = 0, (12, -6)$;
(*c*) $x^2 + y^2 + 8x - 273 = 0, (4, 15)$;
(*d*) $x^2 + y^2 - 4y - 36 = 0, (-2, 8)$.

12. Find the angle of intersection between the line $x + y = 6$ and each of the following circles:

Hint. — The required angle is the angle between the line and the tangent to the circle.

(*a*) $x^2 + y^2 - 4y - 36 = 0$;
(*b*) $x^2 + y^2 + 8x + 8y - 4 = 0$.

13. The point (7, 1) bisects a chord of the circle $x^2 + y^2 = 100$. Find the equation of the chord and its length.

14. The point (4, 7) bisects a chord of the circle $x^2 + y^2 - 4x - 6y - 12 = 0$. Find the equation of the chord and its length.

15. Find the equation of a circle tangent to the line $4x + 3y - 7 = 0$ and having its center at the origin.

General Hint. — In each of the following locus problems, the directions of § 20 as to choice of axes should be followed. After deriving the equation, the locus should be identified and drawn, and its relation to the given data stated.

16. (*a*) The distance between two fixed points is $2c$. Find the locus of a point moving so that the sum of the squares of its distances from the fixed points is $4c^2$.

(*b*) Solve the same problem when the sum of the squares is any constant, as k. *Ans.* The locus is a circle, whose center is midway between the given points and whose radius is $\sqrt{\dfrac{k-2c^2}{2}}$. If $k < 2c^2$, there is no locus.

17. The base of a triangle has the extremities $(a, 0)$ and $(-a, 0)$. Find the locus of the opposite vertex if

(*a*) the vertical angle is 90°;
(*b*) the vertical angle is 45°;
(*c*) the tangent of the vertical angle is any constant k;
(*d*) the median to one of the variable sides is a constant k.

18. The ends of a straight line of constant length a rests on two perpendicular lines. Find the locus of the middle point.

19. Find the locus of a point whose distance from the point $(a, 0)$ is k times its distance from the origin.

20. Find the locus of a point if the sum of the squares of its distances from (*a*) the sides, (*b*) the vertices, of a given square is constant.

21. From the point $(-r, 0)$ chords are drawn in the circle $x^2 + y^2 = r^2$. Find the locus of their mid-points.

22. From one end of a diameter of a circle whose radius is r chords are drawn and produced their own length. Find the locus of the ends of these lines.

51. The Equation of the Circle Derived from Three Conditions. — The correspondence between the geometric conditions necessary to determine a circle and the algebraic conditions necessary to determine its equation is analogous to that discussed for the straight line. The number of independent constants in the general linear equation

$$Ax + By + C = 0$$

is two, for if we divide each term by A, we have the form

$$x + B'y + C' = 0.$$

Geometrically, a straight line is determined by two points, or in general by two geometric conditions. Both forms (13) and (14) of the circle equation involve three arbitrary constants. Geometrically a circle is determined by three points not on a straight line. A circle may be determined in other ways than by the condition of passing through three points, but the determining condition is in any case threefold.

To derive the equation when the geometric conditions are given, it is necessary to express these conditions in three equations involving h, k, and r (or D, E, and F) and solve them simultaneously. The method is illustrated by the following examples.

Example 1. — Find the equation of the circle passing through (1, 7), (8, 6), and (7, −1).

Solution. — Each pair of these coördinates must satisfy the standard equation of the circle (13).

Hence

$$\begin{aligned}(1 - h)^2 + (7 - k)^2 &= r^2,\\ (8 - h)^2 + (6 - k)^2 &= r^2,\\ (7 - h)^2 + (-1 - k)^2 &= r^2.\end{aligned}$$

Solving these equations simultaneously,* we find that $h = 4$,

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* By expanding all three equations and subtracting the first in turn from the second and third, we eliminate h^2, k^2, and r^2, and obtain

$$7h - k = 25$$

and

$$3h - 4k = 0.$$

From these we easily find that $h = 4$ and $k = 3$. Substitution of these values in any one of the three original equations gives $r = 5$.

$k = 3$, and $r = 5$. Therefore the equation is $(x - 4)^2 + (y - 3)^2 = 25$.

We may also solve the problem by using form (14), which gives

$$\begin{aligned} 1 + 49 + D + 7E + F &= 0, \\ 64 + 36 + 8D + 6E + F &= 0, \\ 49 + 1 + 7D - E + F &= 0. \end{aligned}$$

Solving simultaneously, we have

$$D = -8, \qquad E = -6, \qquad \text{and} \qquad F = 0.$$

Then the equation of the circle is $x^2 + y^2 - 8x - 6y = 0$, which may be reduced to $(x - 4)^2 + (y - 3)^2 = 25$ by completing squares.

Example 2. — Find the equation of a circle whose radius is $4\sqrt{5}$ and which is tangent to the line $x - 2y = 20$ at the point $(6, -7)$.

Solution. — In this problem we are given that $r = 4\sqrt{5}$. Let the center C have the coördinates (h, k) as usual. Since the point $(6, -7)$ is on the circle we have at once

$$(6 - h)^2 + (-7 - k)^2 = r^2 = 80. \tag{1}$$

Since the tangent to a circle is perpendicular to the radius drawn to the point of contact, the slope of the line joining C to $(6, -7)$ is the negative reciprocal of the slope of the line $x - 2y = 20$. The former slope is $\frac{k + 7}{h - 6}$ and the latter is $\frac{1}{2}$. Hence

$$\frac{k + 7}{h - 6} = -2. \tag{2}$$

Solving equations (1) and (2) simultaneously gives the values of h and k. From equation (2) we find that

$$k = 5 - 2h.$$

Substituting this in equation (1) and simplifying, we obtain

$$5h^2 - 60h + 100 = 0.$$

Dividing by 5 and factoring, we obtain

$$h = 2 \qquad \text{and} \qquad h = 10.$$

The corresponding values of k are 1 and -15.

There are then two solutions: one in which the center is at (2, 1) and the other in which the center is at (10, −15). The equations of the circles are

$$(x - 2)^2 + (y - 1)^2 = 80$$

or

$$x^2 + y^2 - 4x - 2y - 75 = 0$$

and

$$(x - 10)^2 + (y + 15)^2 = 80$$

or

$$x^2 + y^2 - 20x + 30y + 245 = 0.$$

We can also solve the problem by using the formulas for the distance from a line to a point. By § 42 the distance from the line $x - 2y = 20$ to the center (h, k) is $\dfrac{h - 2k - 20}{-\sqrt{5}}$. This is equal to $+4\sqrt{5}$ or $-4\sqrt{5}$ according to whether the center is above or below the line. Hence

$$h - 2k = 0 \qquad \text{or} \qquad h - 2k = 40.$$

Solving the first equation simultaneously with equation (2), we get the center (2, 1). From the second equation we get the center (10, −15).

An effective graphical check in problems of this kind is to draw the circle with compasses and observe whether or not it satisfies the given conditions.

PROBLEMS

1. Find the equation of the circle passing through the points:

(*a*) $(-1, 2), (4, 2), (-3, 4)$; (*d*) $(-5, 0), (0, 4), (2, 4)$;
(*b*) $(0, 0), (6, 0), (0, -8)$; (*e*) $(2, 0), (0, 0), (-2, -2)$;
(*c*) $(1, 1), (1, 3), (9, 2)$; (*f*) $(6, 0), (-2, 4), (0, -6)$.

2. Find the equation of the circle passing through the points:

(*a*) $(2, 10), (4, 6), (2, 6)$; (*d*) $(4, 8), (-2, 10), (8, -4)$;
(*b*) $(2, 6), (-5, 7), (1, -1)$; (*e*) $(-3, 6), (-2, -6), (2, 4)$;
(*c*) $(3, 3), (-3, 6), (6, 9)$; (*f*) $(5, -6), (3, -4), (1, 2)$.

3. Find the equation of the circle which

(*a*) passes through (2, 0) and (6, 0) and is tangent to the *y*-axis;

(*b*) passes through (3, 2) and (−2, −1) with center on the x-axis;

(*c*) circumscribes the triangle whose sides are the lines $x = 5$, $y = -3$, $x - 2y = 7$;

(*d*) passes through (9, 8) and is tangent to both axes;

(*e*) is tangent to the line $3x - 4y = 2$ at the point (2, 1) and passes through the origin;

(*f*) is tangent to the line $x + 3y = 26$ at the point (8, 6) and has the radius $\sqrt{160}$;

(*g*) passes through (7, −8) and (0, 9) and has its center on the line $x - 2y = 1$;

(*h*) passes through (8, 1) and has its center on the lines $2x - 3y + 26 = 0$ and $3x + 5y = 18$;

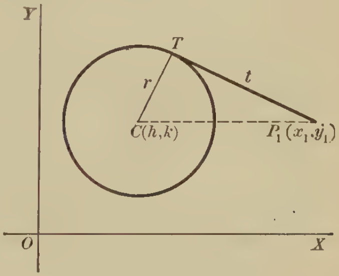
(*i*) is tangent to the circle $x^2 + y^2 = 25$ at the point (4, −3) and has the radius 12;

(*j*) passes through the points (−3, 2) and (11, 0) and has the radius 10;

(*k*) is tangent to the line $x + 2y - 10 = 0$ and passes through the points (3, 3) and (0, 0);

(*l*) has the center (1, 5) and is tangent to the line $3x - 4y + 7 = 0$;

(*m*) is tangent to the lines $2x - 3y = 0$ and $3x + 2y - 13 = 0$ and passes through the point (−4, 6).



52. Length of a Tangent. — Let t be the length of the tangent P_1T, let (h, k) be the coördinates of the center C, and let r be the radius. Then

$$t^2 = CP_1^2 - CT^2.$$

Using the distance formula, this becomes

$$\boldsymbol{t^2 = (x_1 - h)^2 + (y_1 - k)^2 - r^2.} \qquad (15)$$

It is observed that the expression for t is the same in form as the equation of the circle, with the coördinates of P_1 substituted for the variables. Note the similarity of this to the formula for the distance from a line to a point.

Since the right-hand member of (15) is the same in form as the standard equation of the circle, it may also be written

$$t^2 = x_1^2 + y_1^2 + Dx_1 + Ey_1 + F. \qquad (15a)$$

Example. — Find the length of the tangent from (1, 4) to the circle

$$x^2 + y^2 - 4x + 6y - 12 = 0.$$

Also find the point of tangency.

Solution. — Using Formula 15*a*, we have

$$t^2 = 1 + 16 - 4 + 24 - 12 = 25$$

or

$$t = 5.$$

Since the point of tangency is 5 units distant from (1, 4), it lies on the circle:

$$(x - 1)^2 + (y - 4)^2 = 25$$

or

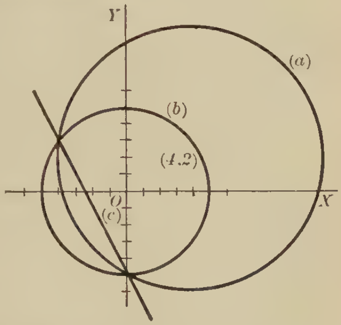
$$x^2 + y^2 - 2x - 8y - 8 = 0.$$

But it also lies on the given circle:

$$x^2 + y^2 - 4x + 6y - 12 = 0.$$

Solving these equations simultaneously, we obtain the solutions (5, 1) and (−2, 0) corresponding to the two possible tangents.

53. Common Chord of Two Circles. — To find the common chord of two circles we may solve the equations simultaneously, to get the points of intersection, and then use the two-point form of the line equation. A better method will now be illustrated.



Let the two circles have the equations

$$x^2+y^2-8x-4y-44=0, \quad (a)$$

and

$$x^2+y^2-25=0. \quad (b)$$

Subtracting,

$$8x+4y+19=0. \quad (c)$$

By the theorem of § 45 the locus of (c) contains the points common to (a) and (b). Since it is of the first degree, it defines a straight line. Hence this line is the common chord.

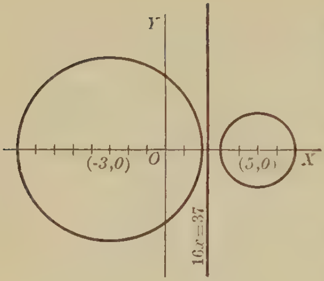
The equation of the common chord of two circles can always be obtained by subtracting the equation of one of them from that of the other, as above. For, since nothing above the first degree appears in the equation of a circle except the characteristic terms $x^2 + y^2$, which are always present, these will be cancelled by subtraction, leaving a linear equation whose locus contains the points of intersection of the two circles.

Exercise 2. Derive the general equation of the common chord of two circles.

Ans. $(D - D')\,x + (E - E')\,y + (F - F') = 0.$

Exercise 3. Prove that the common chord of two circles is perpendicular to their line of centers.

Hint. — The line of centers of two circles is determined by the points (h, k) and (h', k'). Hence its slope is $\dfrac{k - k'}{h - h'}$. Compare this with the slope of the common chord found in Exercise 2 and express the result in terms of h, k, h', and k'.



54. Radical Axis. — Consider the following two examples.

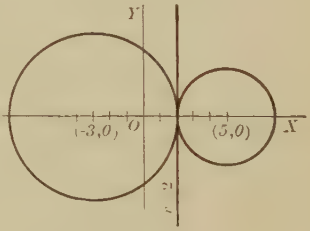
Example 1. — Find the equation of the common chord of the circles

$$(x+3)^2+y^2=25$$

and $$(x-5)^2+y^2=4$$

and the intersections of the circles.

Applying § 53, the equation is found to be $16x = 37$; but the points of intersection are imaginary, as is seen by inspection of the equations, since the sum of the radii is 7, while the distance between the centers is $3 + 5 = 8$.



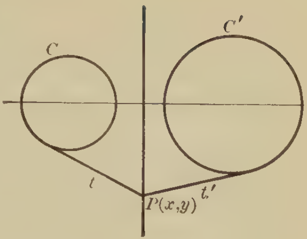
Example 2. — Find the equation of the common chord of the circles

$$(x+3)^2+y^2=25$$

and $$(x-5)^2+y^2=9.$$

Here the distance between the centers is the same as the sum of the radii and the circles are tangent to each other at the point (2, 0). Also we have by subtraction $x = 2$, whose locus, passing through the point (2, 0) and being perpendicular to the line of centers, is the common tangent.

Thus we see that the application of the rule for finding a common chord gives a common chord when the two circles intersect, a common tangent when they are tangent, and finally, when they do not meet, a line which has no apparent significance. We shall find, however, that the line in all three cases has a common property; viz., it is the locus of the point from which tangents to the two circles are equal.



To prove this, let t and t' be equal tangents from P to the circles C and C' respectively, of which the equations are

$$x^2 + y^2 + Dx + Ey + F = 0$$

and $$x^2 + y^2 + D'x + E'y + F' = 0.$$

From § 52, Formula 15*a*,

$$t^2 = x^2 + y^2 + Dx + Ey + F.$$

Also $$t'^2 = x^2 + y^2 + D'x + E'y + F'.$$

Hence

$$t^2 - t'^2 = (D - D')\,x + (E - E')\,y + F - F'.$$

By hypothesis, $t = t'$, or $t^2 - t'^2 = 0$.

Therefore,

$$(D - D')\,x + (E - E')\,y + F - F' = 0 \qquad (16)$$

is the equation of the locus of P.

But this is the equation obtained by eliminating the terms in x^2 and y^2. Hence we have the theorem:

The locus of the point from which the tangents to two circles are equal is the line whose equation is obtained by eliminating the terms in x^2 *and* y^2 *between the equations of the circles.*

This locus is called the *radical axis.* When the circles intersect, it is their common chord; and when they touch, it is their common tangent.

55. Systems of Circles. — The geometric constants of the circle designated by h, k, and r determine the center and radius. If two of these constants are restricted by assigning conditions to them, numerical or otherwise, the equation will represent a *system of circles* defined more or less completely according to the nature of the conditions.

Thus, if $h = 2$ and $k = 4$, we have a system of concentric circles of varying radius. Again, if $h = k$ and $r = 10$, we have a system of equal circles whose centers lie on the line $x - y = 0$.

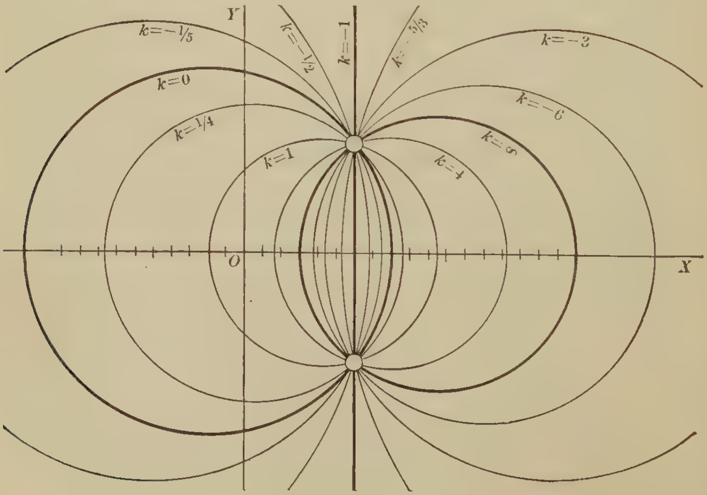
Systems having the Same Radical Axis. — If we use the theorem of § 45, we find that $C + k \cdot C' = 0$, where k is any constant and $C = 0$ and $C' = 0$ are the equations of two circles, will define a system of circles passing through the points of intersection of C and C'. But the radical axis of any two circles passes through the points of intersection, real or imaginary; hence this equation defines a system of circles having the same radical axis.

The figure below illustrates the system of circles having the same radical axis as

$$x^2 + y^2 + 4x - 96 = 0$$

and

$$x^2 + y^2 - 21x + 54 = 0.$$



These meet at the points (6, 6) and (6, −6); the equation of the radical axis is $x = 6$. The equation of the system is

$$x^2 + y^2 + 4x - 96 + k(x^2 + y^2 - 21x + 54) = 0.$$

In the figure the circles drawn in heavy lines are the given circles and the others are the circles corresponding to

$$k = \tfrac{1}{4},\ 1,\ 4,\ -\tfrac{1}{5},\ -\tfrac{1}{2},\ -1,\ -\tfrac{5}{3},\ -3,\ -6.$$

Note that when k approaches −1, the circle becomes of infinite size and approaches as a limit the radical axis.

PROBLEMS

1. Find the least distance from the point (6, 8) to each circle of Problem 2, page 87.

2. Find the length of the tangent from the point (6, 8) to each circle of Problem 2, page 87.

3. Find the coördinates of the points of contact of the tangents from the given point to the given circle:

(*a*) $x^2 + y^2 - 6x - 6y - 7 = 0,\ (-4, 4)$;
(*b*) $x^2 + y^2 = 17,\ (5, 3)$.

4. Find the equations of the radical axes and the points of intersection of the following pairs of circles:

(*a*) $x^2 + y^2 + 4x + 6y = 7,$
$x^2 + y^2 - 2x - 6y + 5 = 0;$
(*b*) $x^2 + y^2 - 10x - 10y = -44,$
$x^2 + y^2 - 36 = 0;$
(*c*) $x^2 + y^2 + 8x + 18y = 1,$
$3x^2 + 3y^2 + 20x + 44y = 11;$
(*d*) $x^2 + y^2 + 4x + 3y - 32 = 0,$
$x^2 + y^2 + 3x + 5y - 28 = 0;$
(*e*) $x^2 + y^2 - 2x + 4y - 4 = 0,$
$2x^2 + 2y^2 - 4x + 6y - 7 = 0;$
(*f*) $x^2 + y^2 + 7x - 5y = 8,$
$x^2 + y^2 + 8x - 4y = 5;$

(*g*) $x^2 + y^2 = 25,$
$x^2 + y^2 + 2x - 4y = 55;$
(*h*) $x^2 + y^2 - 20x - 20y = 0,$
$x^2 + y^2 + 20x + 200 = 0.$

5. Find a point from which equal tangents may be drawn to each of the following circles:

(*a*) $x^2 + y^2 + 3x + 2y = 6,$
$x^2 + y^2 - 3x - 2y = -2,$
$x^2 + y^2 - 4x - 8y = -12;$
(*b*) $x^2 + y^2 + 4x - 3y + 5 = 0,$
$2x^2 + 2y^2 + 5x - 6y + 1 = 0,$
$x^2 + y^2 + 5x - 4y + 5 = 0.$

6. In each pair of circles in Problem 4 find the line of centers, and prove that it is perpendicular to the radical axis.

7. Prove that the radical axes of any three circles taken in pairs meet in a point or are parallel.

8. What is the equation of the system of circles whose center is (8, 3)? Which one of these circles passes through the point (−8, 0)?

9. Find the equation of the circle which passes through the points of intersection of the circles $x^2 + y^2 + 4x + 3y = 32$ and $x^2 + y^2 + 3x + 5y = 28$ and the point (2, 2).

Hint. — First write the equation of the system of circles passing through the two points of intersection.

10. Find the equation of the circle whose center is on the x-axis and which passes through the points of intersection of the circles in Problem 9.

Hint. — If the center is on the x-axis, what must the coefficient of the first degree term in y be?

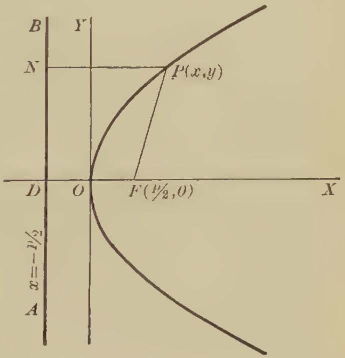
CHAPTER V

THE PARABOLA

56. Definition. — *The parabola is the locus of a point such that its distance from a fixed point is equal to its distance from a fixed line.*

The fixed point is called the *focus* and the fixed line the *directrix*. The line through the focus perpendicular to the directrix is called the *principal axis*. The point on the principal axis midway between the focus and the directrix lies on the locus and is called the *vertex* of the parabola.

57. The Equation of the Parabola. — Let AB be the directrix, F the focus, DF the principal axis, and O the vertex of the parabola.



To derive the equation take the principal axis as the x-axis and the vertex as the origin. Call the distance between the directrix and the focus p. Then the coördinates of F are $\left(\frac{p}{2}, 0\right)$, and the equation of the directrix is $x = -\frac{p}{2}$.

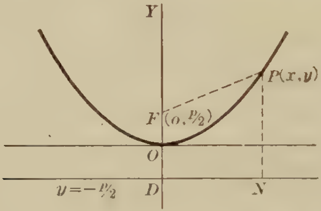
Let $P(x, y)$ be any point on the locus. Draw NP perpendicular to AB, and join FP. By definition $FP = NP$,

that is,
$$\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = x + \frac{p}{2}.$$

Squaring and reducing, we have

$$y^2 = 2\,px. \tag{17}$$

This is the standard equation of the parabola when the vertex is taken as the origin and the principal axis as the x-axis. If the principal axis is taken as the y-axis and the vertex as the origin, the variables are interchanged and the equation is



$$x^2 = 2\,py. \tag{17a}$$

If the focus is in a negative direction from the vertex, the right-hand members of these equations have minus signs.

The simplicity of these equations is due entirely to the choice of the coördinate axes. If the axes are chosen in any other way, a different and more complicated equation is obtained, but the curve itself is unaltered.* Therefore we shall study the properties of the parabola by means of the simple forms and reserve the general equations for a later chapter.

Exercise 1. Derive equation (17*a*) from a figure.

Exercise 2. Derive equations (17) and (17*a*) when the focus is in a negative direction from the vertex.

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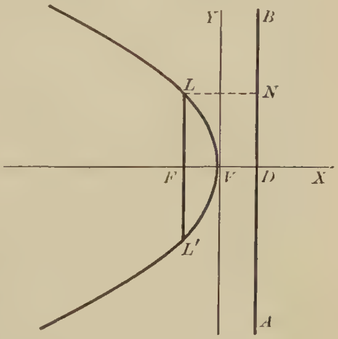
* The student should review Example 2, § 20, and Problem 2, page 31, to realize the importance of this statement.

58. Discussion of the Parabola. — The form of equation (17) shows that the parabola is symmetrical with respect to the principal axis, and that the parabola crosses this axis at the vertex only.

Since negative values of x make y imaginary, the curve lies wholly to the right of the vertex and has no point nearer the directrix. It recedes indefinitely from both axes, since $y = \pm\sqrt{2\,px}$ increases indefinitely with x. A different parabola is obtained for each value assigned to p.

59. Latus Rectum. — A line joining the focus F to any point of the parabola is a *focal radius;* a chord of the parabola passing through F is a *focal chord.* The focal chord $L'L$ parallel to the directrix is called the *latus rectum.* By definition of the parabola, the distance of L from the focus is the same as that from the directrix.



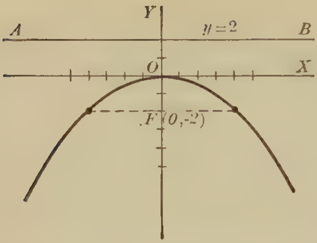
Hence

$$FL = LN = FD = p$$

and the length of the whole latus rectum is $2\,p$.

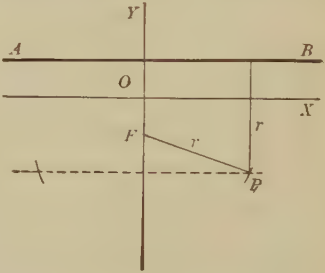
60. Construction of the Parabola. — Consider the equation $x^2 = -8\,y$. This is of the form $x^2 = -2\,py$, and hence is the equation of a parabola. To draw the curve we can make a table of values as usual, or proceed according to one of the three following methods.

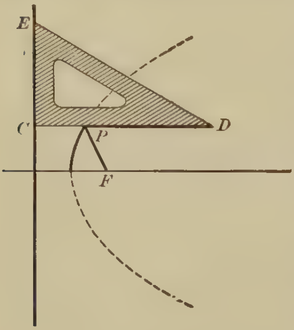
1. *Sketch.* — From the form of the equation, $2\,p = 8$, or $p = 4$. Since the sign before $2\,p$ is minus, the curve lies wholly below the x-axis.



Hence the focus is $\left(0, -\frac{p}{2}\right)$, or $(0, -2)$, and the equation of the directrix is $y = 2$. Measuring off $p = 4$ to the right and left of the focus gives the ends of the latus rectum. These together with the vertex O make three points on the curve, which suffice for a sketch, as the general shape of a parabola is known.

2. *Ruler and Compasses.* — If a more accurate graph is desired, find the focus and directrix as above. Then with the focus as a center and a radius $r > \frac{p}{2}$ draw an arc. This will cut the line parallel to the directrix and r units from it in two points which lie on the parabola by definition. By using coördinate paper and varying r, as many points as are desired may be readily constructed.





3. *Mechanical Construction.* — Place a right triangle with one leg CE on the directrix. Fasten one end of a string whose length is CD at the focus F and the other end to the triangle at D. With a pencil at P keep the string taut. Then $FP = CP$; and as the triangle is moved along the directrix the point P will describe a parabola.

61. Properties and Applications. — The parabola (and other curves discussed later) was discovered by the Greek geometers, who also developed many of its geometric properties. In addition to those given in this chapter the following may be mentioned. The section of a right circular cone by a plane parallel to an element is a parabola. (See Appendix.) The mid-points of all chords having the same slope are on a line parallel to the principal axis. (See § 144.) The area of the segment formed by a parabola and any chord is equal to two-thirds the area of the parallelogram having the given chord for one side and the opposite side tangent to the parabola.

The importance of the curve to-day is due chiefly to its scientific applications, which in turn are a consequence of its purely mathematical properties. Unfortunately a complete discussion of most of these applications cannot be given without the aid of the calculus, but it may be of interest at this point to state a few without proof.

If a parabola is rotated about its axis a parabolic surface is formed. Rays of light parallel to the axis and meeting the concave side of this surface are reflected so as to pass through the focus. Conversely, if a source of light be placed at the focus all rays which meet the surface will be reflected parallel to the axis. It is for this reason that parabolic reflectors are used in searchlights. (See § 142.)

A projectile which is fired at any elevation traces a path which is approximately a parabola. If there were no atmosphere to retard the progress of the projectile the path would be a true parabola. (See § 126.)

Several different types of bridges involve the use of a parabolic arch, with the vertex either above or below the roadbed of the bridge. The cable of a suspension bridge whose total weight is uniformly distributed over the length of the bridge also takes the form of a parabola.

PROBLEMS

1. From each of the following equations, find the coördinates of the focus, the equation of the directrix, and the length of the latus rectum, and draw the parabola to scale:

(*a*) $y^2 = 12x$;
(*b*) $x^2 = 6y$;
(*c*) $x^2 = -6y$;
(*d*) $3y^2 + 8x = 0$;
(*e*) $y^2 = -24x$;
(*f*) $x = 4y^2$;
(*g*) $x^2 = 16y$;
(*h*) $x^2 + 12y = 0$;
(*i*) $x^2 = 4ay$;
(*j*) $x = ky^2$.

2. The same as Problem 1 for the following equations;

(*a*) $y^2 = 8x$;
(*b*) $y^2 = -8x$;
(*c*) $x^2 = 15y$;
(*d*) $3x^2 + 14y = 0$;
(*e*) $y^2 + 4ax = 0$;
(*f*) $y = kx^2$;
(*g*) $y^2 + 10x = 0$;
(*h*) $y^2 - 9x = 0$;
(*i*) $x^2 + 20y = 0$;
(*j*) $x^2 = 3y$.

3. Write the equations of parabolas satisfying the following data:

(*a*) directrix $y = -5$, focus $(0, 5)$;
(*b*) directrix $x = 6$, focus $(-6, 0)$;
(*c*) vertex $(0, 0)$, focus $(a, 0)$;
(*d*) vertex $(0, 0)$, one point of curve $(3, -4)$;
(*e*) vertex $(0, 0)$, directrix $x = -4$;
(*f*) vertex $(0, 0)$, focus $(-3, 0)$.

4. Write the equations of the parabolas which have:

(*a*) directrix $y = 6$, focus $(0, -6)$;
(*b*) directrix $x = -b$, vertex $(0, 0)$;
(*c*) vertex $(0, 0)$, focus $(-4, 0)$;
(*d*) vertex $(0, 0)$, one point of curve $(-6, 8)$;
(*e*) directrix $x = 8$, focus $(-8, 0)$;
(*f*) vertex $(0, 0)$, focus $(0, 7)$.

5. Find the locus of a point which moves so that its distance from the point $(0, 6)$ is three units greater than its distance from the line $y + 3 = 0$.

6. Find the locus of a point which moves so that its distance from the point $(-5, 0)$ is two units less than its distance from the line $x = 7$.

7. The base of a triangle is fixed with its extremities at $(\pm a, 0)$. Find the locus of the opposite vertex if the ratio of the base to the altitude is constantly equal to the slope of the median on the fixed base.

8. Two vertices of a triangle are $A\ (2c, c)$ and $B\ (-2c, c)$. Find the locus of the third vertex P if the slope of PB is one less than the slope of PA.

9. Find the equation of the circle circumscribing the portion of the parabola $y^2 = 2px$ cut off by the latus rectum.

10. Find the locus of the center of a circle tangent to a given line and passing through a fixed point at a distance of $2c$ from the given line.

11. In the parabola $x^2 = 2py$ an equilateral triangle is inscribed with one vertex at the origin. Find the length of a side.

12. Show that the latus rectum of any parabola is a third proportional to the abscissa and ordinate of any point upon it.

13. Find the length of the focal radius to the point (a, b) on the parabola $y^2 = 2px$.

14. Find the area of the isosceles right triangle inscribed in the parabola $x^2 = 2py$ with the vertex of the right angle at the origin.

15. Find the points of intersection of the parabolas $y^2 = 2px$ and $x^2 = 2py$.

16. The focal radius of a given point on the parabola $x^2 = 2py$ is the same length as the latus rectum. Find the coördinates of the point.

17. In Problem 14, find the length of the hypotenuse of the right triangle.

18. In Problem 11, find the area of the given triangle.

19. The parabola $y^2 = -2px$ has one end of a focal chord at the point $(-\frac{1}{2}, 2)$. Find the coördinates of the other end.

20. The parabola $x^2 = 2py$ has one end of a focal chord at the point $(6a, 9a)$. Find the coördinates of the other end.

21. Given the parabola $x^2 = -12y$. Find the equation of the chord joining the points whose abscissas are 6 and 10.

22. Given the parabola $y^2 = 8x$. Find the equation of the focal chord through the point whose ordinate is 4.

23. Find the locus of the mid-points of the ordinates of the parabola $y^2 = 2px$.

24. Show that the lines drawn from the intersection of the directrix and the principal axis to the extremities of the latus rectum are perpendicular to each other.

25. Find the locus of the ends of the latus rectum of the parabola $x^2 = 2py$ when p is allowed to vary.

26. Show that the circle $x^2 - 2ax + y^2 = 0$ and the parabola $y^2 = 2px$ meet in but one point unless $a > p$.

27. From the definition of the parabola, derive its equation when:

(*a*) the x-axis is the principal axis with the origin at the focus;

(*b*) the y-axis is the principal axis with the origin on the directrix.

28. The same as Problem 27 when:

(*a*) the y-axis is the directrix and the x-axis is the principal axis.

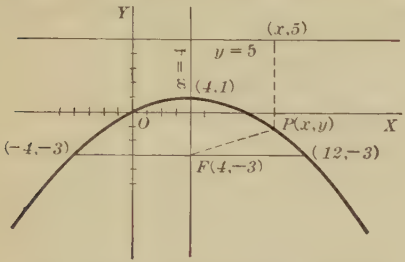
(*b*) the y-axis is the principal axis with the origin at the focus.

29. Locate the principal axis, the vertex, and the extremities of the latus rectum for each parabola. Sketch the curve.

(*a*) directrix $y = 5$, focus $(4, -3)$;

Solution. — Since the principal axis is perpendicular to the directrix and passes through the focus, it is the line $x = 4$.

The vertex, midway between the focus and directrix, is the point (4, 1). In this case $p = 8$, and the extremities of the latus rectum are (12, −3) and (−4, −3).



(b) directrix $x + 3 = 0$, focus $(-7, 2)$;

(c) directrix $y + 2 = 0$, focus $(3, 6)$;

(d) directrix $x = h - a$, focus $(h + a, k)$.

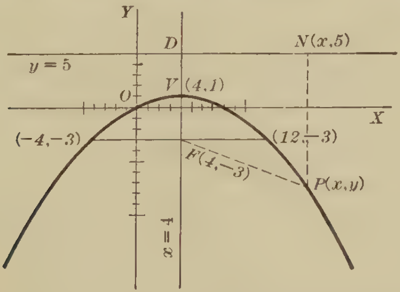
30. Locate the directrix, the focus, and the extremities of the latus rectum for each of the parabolas. Sketch the curve. (How many solutions in each case?)

(a) principal axis $x = 3$, vertex $(3, 1)$, $p = 10$;

(b) principal axis $y = -2$, vertex $(1, -2)$, latus rectum 18;

(c) principal axis $y = a$, vertex $(0, a)$, distance between vertex and focus b.

31. Derive the equation of each parabola in Problem 29.



Solution. — (a) If $P(x, y)$ is any point on the parabola, its distance from the focus $(4, -3)$ is $\sqrt{(x-4)^2 + (y+3)^2}$, its distance from the directrix $y = 5$ is $y - 5$. Hence:

$\sqrt{(x-4)^2 + (y+3)^2} = y - 5$. Squaring and collecting terms, we have $x^2 - 8x + 16y = 0$.

32. Derive the equation of each parabola in Problem 30.

33. Derive the equations of the following parabolas, find the principal axis and the length of the latus rectum, and sketch the curve.

(*a*) directrix $x + 4 = 0$, vertex $(1, -2)$;
(*b*) directrix $y = -1$, vertex $(-1, -4\frac{1}{2})$;
(*c*) directrix $x = 0$, focus $(-4, 3)$;
(*d*) vertex (a, b), focus $(a, 2b)$;
(*e*) directrix $x = a$, vertex $(a + p/2, c)$.

34. The same as Problem 33 for the following conditions:

(*a*) vertex $(-1, 4)$, focus $(-4, 4)$;
(*b*) directrix $x + 6 = 0$, vertex $(-2, 2)$;
(*c*) directrix $y = 3$, focus $(2, -5)$;
(*d*) directrix $y = k - p/2$, vertex (h, k);
(*e*) directrix $x = a$, vertex $(0, b)$.

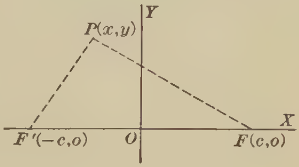
CHAPTER VI

THE ELLIPSE

62. Definition. — *The ellipse is the locus of a point such that the sum of its distances from two fixed points is constant.*

The two fixed points are called the *foci*. In the following we denote the distance between the foci by $2c$ and the sum of the distances of any point on the ellipse from the foci by $2a$. Obviously $2c < 2a$, or $c < a$.

63. The Equation of the Ellipse. — Take the origin midway between the foci F' and F, and let the line joining these two points be the x-axis. Then the coördinates of the foci are $(\pm c, 0)$, and any point $P(x, y)$ on the ellipse must satisfy the relation



$$F'P + FP = 2a$$

or $$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

Transpose the second radical, square both sides and collect terms. We get

$$4cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2}.$$

Solving for the radical gives

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x.$$

Squaring again and collecting terms, we have

$$\frac{a^2 - c^2}{a^2}x^2 + y^2 = a^2 - c^2.$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Since $c < a$, $a^2 - c^2$ has a positive value for every ellipse, and if we set $b^2 = a^2 - c^2$, then b will be a real number. The equation of the ellipse then becomes

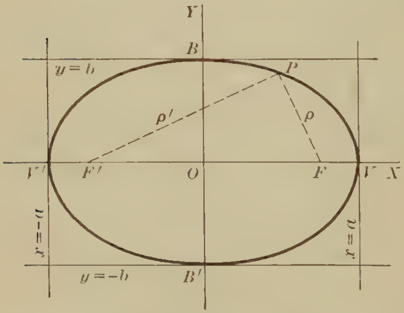
$$\frac{\boldsymbol{x}^2}{\boldsymbol{a}^2} + \frac{\boldsymbol{y}^2}{\boldsymbol{b}^2} = \mathbf{1} \qquad \textbf{(18)}$$

or

$$b^2x^2 + a^2y^2 = a^2b^2.$$

64. Discussion of the Ellipse. — The locus of equation (18) has intercepts $\pm a$ on the x-axis, and $\pm b$ on the y-axis. It is symmetrical with respect to both axes and hence is symmetrical with respect to the origin O.

Solving for y, we have $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$, a form which shows that all values of x numerically greater than a must be excluded. Similarly all values of y numerically greater than b must be excluded. Hence the ellipse lies wholly within the rectangle whose sides are $x = \pm a$, $y = \pm b$.



The line joining the foci $F'F$ is called the *principal axis*, and the points V and V' where the principal axis intersects the ellipse are called the *vertices.* The segment $V'V$ of the principal axis intercepted by the ellipse is called the *major axis.* The segment $B'B$ of the

y-axis is called the *minor axis.* The major axis is of length $2a$ and the minor axis of length $2b$. O is called the center of the ellipse and is the mid-point of every chord of the ellipse which passes through it, since it is the center of symmetry. From the definition of b (§ 63) we have

$$a^2 = b^2 + c^2 \tag{19}$$

a relation connecting the semi-major axis, the semi-minor axis, and the distance of either focus from the center.

Exercise 1. Show that the distance from either focus to one end of the minor axis is a.

Exercise 2. Show that the semi-minor axis is the mean proportional between the segments into which either focus divides the major axis.

65. Focal Radii. — The distances of any point on the ellipse from the foci are called the *focal radii* of the point. Denoting these by ρ and ρ' we have from the definition of the ellipse,

$$\rho + \rho' = 2a. \tag{20}$$

Hence *the sum of the focal radii to any point on the ellipse is equal to the major axis.*

66. Latus Rectum. — The chord through either focus perpendicular to the major axis is called the latus rectum. If in the equation of the ellipse we put $x = \pm c$, then

$$\frac{c^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{y^2}{b^2} = \frac{a^2 - c^2}{a^2} = \frac{b^2}{a^2}.$$

Therefore

$$\frac{y}{b} = \pm\frac{b}{a}, \qquad \text{or} \qquad y = \pm\frac{b^2}{a}.$$

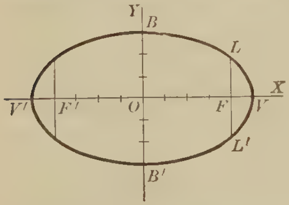
Hence the length of the latus rectum is $\frac{2b^2}{a}$.

As in the case of the parabola, the location of the ends of the latus rectum is a considerable aid to accuracy in sketching an ellipse of given dimensions.

Exercise 3. Show that the latus rectum of an ellipse is a third proportional to the major and the minor axes.

PROBLEMS

1. Find the semi-major axis, the semi-minor axis, the latus rectum, the foci, and the vertices of the following ellipses, and sketch each curve:



(*a*) $9x^2 + 25y^2 = 225$.

Solution. — Dividing by 225, we have

$$\frac{x^2}{25} + \frac{y^2}{9} = 1,$$

which is of the standard form. Hence $a = 5$, and $b = 3$. By Formula 19, $c = 4$, and by § 66 the latus rectum is $3\frac{3}{5}$. The foci are $(\pm 4, 0)$ and the vertices are $(\pm 5, 0)$. The ends of the minor axis are $(0, \pm 3)$ and the value of the latus rectum gives four additional points, *viz.*, $\left(\pm 4, \pm\frac{9}{5}\right)$. These are ample for a sketch of the curve.

(*b*) $9x^2 + 36y^2 = 324$;
(*c*) $4x^2 + 9y^2 = 36$;
(*d*) $9x^2 + 16y^2 = 16$;
(*e*) $5x^2 + 30y^2 = 150$;
(*f*) $x^2 + 9y^2 = 25$;
(*g*) $25x^2 + 49y^2 = 196$;
(*h*) $x^2 + 4y^2 = 64$;
(*i*) $x^2 + 2y^2 = 50$.

2. The same as Problem 1 for the following ellipses:

(*a*) $9x^2 + 16y^2 = 144$;
(*b*) $4x^2 + 25y^2 = 100$;
(*c*) $16x^2 + 36y^2 = 225$;
(*d*) $x^2 + 4y^2 = 9$;
(*e*) $9x^2 + 49y^2 = 256$;
(*f*) $3x^2 + 10y^2 = 30$;
(*g*) $x^2 + 4y^2 = 36$;
(*h*) $4x^2 + 25y^2 = 64$;
(*i*) $x^2 + 5y^2 = 45$;
(*j*) $4x^2 + 9y^2 = 144$.

3. Write the equation of the ellipse with the center at the origin, given:

(*a*) semi-major axis 10, one focus (8, 0);
(*b*) one vertex $(2\sqrt{3}, 0)$, semi-minor axis 2;
(*c*) one vertex (6, 0), latus rectum $6\frac{2}{3}$;
(*d*) one end of semi-minor axis (0, 4), latus rectum 4;
(*e*) one focus (−12, 0), minor axis 10.

4. Write the equation of the ellipse with the center at the origin and having the principal axis for the x-axis, given:

(*a*) $a = 13$, $b = 5$;
(*b*) $b = 3$, $c = 3$;
(*c*) $a = 2\sqrt{5}$, $c = 4$;
(*d*) $a = 8$, latus rectum $6\frac{1}{4}$;
(*e*) $b = 3\sqrt{2}$, latus rectum 5;
(*f*) $c = 2\sqrt{2}$, latus rectum 4.

5. Find the relation between a and b if the latus rectum is equal to:

(*a*) the semi-major axis; (*b*) the semi-minor axis.

6. If the latus rectum is equal to the distance between the foci, find the relation between a, b, and c.

7. Find the locus of a point which moves so that the sum of its distances from (±6, 0) is 20.

8. Find the locus of the vertex of a triangle with a fixed base of length 16 and such that the product of the slopes of the variable sides is $-\frac{25}{64}$.

9. Given the circle $x^2 + y^2 = 36$. From points on the circumference, perpendiculars are drawn to the horizontal diameter. Find the equation of the locus of the mid-points of these perpendiculars.

10. In the circle of Problem 9 perpendiculars are drawn from the vertical diameter to the circumference and each perpendicular is extended its own length beyond the circumference. Find the locus of the extremities of these lines.

11. A point moves so that it is two-thirds as far from the point (4, 0) as it is from the line $x = 9$. Find the locus.

12. A point moves so that its distance from the point (6, 0) is three-fourths its distance from the line $x = 10\frac{2}{3}$. Find the locus.

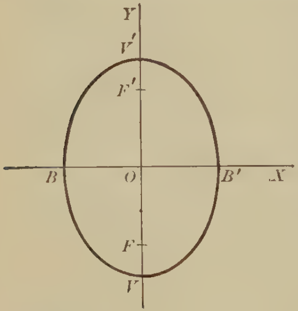
13. The base of a triangle has its extremities at $(\pm a, 0)$. Find the locus of the opposite vertex if the product of the tangents of the base angles is b^2/a^2.

14. Find the equation of an ellipse with the center at the origin, and the foci midway from the center to the vertices.

15. Show that the equation $x^2 + 4y^2 = k$ represents a system of ellipses in each of which the major axis is twice the minor axis, if k is positive. Discuss the cases that k is zero or negative.

16. Write the equation of an ellipse whose major axis is three times the minor axis and which passes through the point $(6, -2)$.

67. Ellipse with Foci on *Y*-axis. — If the foci are on the y-axis, this becomes the principal axis. Calling the major axis and minor axis of the ellipse $2a$ and $2b$ respectively as before, the effect on the equation of the ellipse is merely to interchange the variables x and y. It is then



$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, \qquad (18a)$$

or $$b^2y^2 + a^2x^2 = a^2b^2.$$

In this form it is readily seen that the coördinates of the vertices are $(0, \pm a)$, and the coördinates of the foci are $(0, \pm c)$.

Exercise 4. Prove as in § 63 that the central equation of an ellipse having the foci on the y-axis is $b^2y^2 + a^2x^2 = a^2b^2$.

68. Limiting Forms of the Ellipse. — If $c = 0$ the two foci F' and F coincide at the center, and $b = a$. The equation of the ellipse is then $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$, or $x^2 + y^2 = a^2$, the locus of which is a circle with the center at the origin and the radius a. Hence the circle may be considered as a limiting form of an ellipse whose foci coincide at the center.

On the other hand, if $c = a$ then $b = 0$, and the two foci F' and F coincide with the vertices V' and V respectively. As c approaches a the ellipse "flattens out" and approaches the line segment $V'V$. Hence a line segment may be considered as a limiting form of an ellipse whose foci are at the extremities of the segment.

69. Eccentricity. — From the previous section it is evident that the shape of an ellipse depends on the relative values of c and a. The fraction $\frac{c}{a}$ is called the *eccentricity* and is denoted by e.

Hence $$c = ae \tag{21}$$

and since c cannot be greater than a by § 62, the eccentricity of an ellipse will have some value between 0 and 1. If we take a fixed line segment $V'V$ as the major axis of an ellipse and let e vary from 0 to 1, the ellipse will vary accordingly from the circle having $V'V$ as a diameter to the line segment $V'V$.

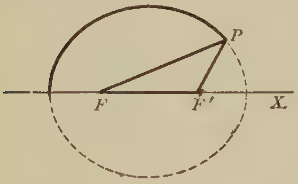
Exercise 5. If $P(x, y)$ is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ [illegible] show that the focal radii drawn to P are of length $a \pm ex$ respectively.

Hint. — Use the distance formula and eliminate y.

70. Construction of an Ellipse. — When a careful drawing of an ellipse is required, it is always possible to form the equation and compute a table of values. This, however, entails considerable labor, and it is sometimes better to proceed according to one of the following methods.

1. *Construction by Points.* Lay off the major axis on coördinate paper. Then with a radius $\rho < 2a$ and a focus as a center draw an arc. With the other focus as a center and $\rho' = 2a - \rho$ as a radius, draw another arc cutting the first in two points. These are points of the ellipse by § 65. By varying ρ we get as many points as are desired.

2. *Mechanical Construction.* The relation of the focal radii may be used to construct the ellipse mechanically. On a drawing board fasten two tacks at the foci F and F', and tie a string about them of length equal to $2a + 2c$. If a pencil is placed in the loop FPF' and moved so as to keep the string taut, then $PF + PF'$ is a constant equal to $2a$, and P describes an ellipse (§ 62).



71. Applications. — Like the parabola the ellipse is a section of a right circular cone, an ellipse being formed if the cutting plane cuts all the elements of the cone (see Appendix). It is also the curve formed when a right circular cylinder is cut by a plane not parallel or perpendicular to the base (see Problem 10, below). Among the applications of the ellipse in the arts and sciences the following may be mentioned.

If the focal radii are drawn to any point of an ellipse they make the same angle with the ellipse. (See § 143.) Hence if a surface is formed by rotating an ellipse about its major axis, any rays of light, or sound waves, proceeding from either focus will be reflected from the surface so as to converge at the other focus. This is the principle of the so called "whispering galleries."

The paths of the planets in their motion around the sun are ellipses with one focus at the sun. The earth's orbit is an ellipse with an eccentricity of approximately $\frac{1}{60}$.

In the construction of bridges, elliptic arches are used as well as parabolic arches. They are also used in architecture.

PROBLEMS

1. Find the vertices, foci, eccentricity, and length of the latus rectum of the ellipses and sketch the curve.

(*a*) $25x^2 + 9y^2 = 225.$

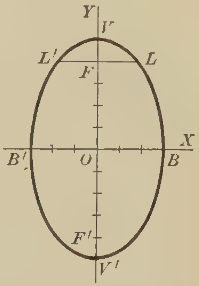
Solution. — Dividing by 225, we have

$$\frac{x^2}{9} + \frac{y^2}{25} = 1.$$

Since a is always $> b$, $a = 5$ and $b = 3$, and the equation is of the form

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

Hence the vertices are $(0, \pm 5)$, the foci $(0, \pm 4)$, $e = \frac{4}{5}$, and the latus rectum $\frac{18}{5}$.



(*b*) $36x^2 + 4y^2 = 144;$

(*c*) $16x^2 + 9y^2 = 100;$

(*d*) $9x^2 + 4y^2 = 49;$

(*e*) $17x^2 + 8y^2 = 136;$

(*f*) $9x^2 + 25y^2 = 36;$

(*g*) $4x^2 + y^2 = 4.$

2. Investigate the following equations as in Problem 1.

(*a*) $9x^2 + 4y^2 = 144;$

(*b*) $25x^2 + 4y^2 = 100;$

(*c*) $9x^2 + 49y^2 = 196;$

(*d*) $5x^2 + 2y^2 = 30;$

(*e*) $64x^2 + 25y^2 = 100;$

(*f*) $25x^2 + 16y^2 = 25.$

3. Write the equation of the ellipse with the center at the origin, given:

(*a*) vertex $(0, 4)$, focus $(0, 3)$;
(*b*) latus rectum $\frac{50}{13}$, vertex $(0, -13)$;
(*c*) one end of minor axis $(2\sqrt{2}, 0)$, focus $(0, -4)$;
(*d*) vertex $(0, 5)$, eccentricity $\frac{3}{5}$;
(*e*) one end of minor axis $(5, 0)$, eccentricity $\frac{12}{13}$.

4. Write the equations of the following ellipses with the center at the origin, the foci being first on the x-axis and second on the y-axis:

(*a*) $a = 8, b = 6$;
(*b*) $b = 12, c = 5$;
(*c*) $c = 6, e = \frac{1}{2}$;
(*d*) $a = 10$, latus rectum 8;
(*e*) $c = 5, a = 3\sqrt{5}$;
(*f*) $a = 3\sqrt{2}, e = \frac{2}{3}$;
(*g*) $b = 4$, latus rectum 6;
(*h*) $b = 5, e = \frac{4}{5}$;
(*i*) $c = 2\sqrt{5}$, latus rectum $5\frac{1}{3}$;
(*j*) $c = \frac{1}{2}\sqrt{3}$, latus rectum $2\sqrt{3}$.

5. Express b in terms of a and e.

6. Express b in terms of c and e.

7. Find the locus of a point moving so that the sum of its distances from the points $(0, \pm 8)$ is 24.

8. Find the locus of the mid-points of the ordinates of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

9. The ordinates of the ellipse $a^2x^2 + b^2y^2 = a^2b^2$ are each extended their own length. Find the locus of their extremities.

10. Prove that the projection of a circle upon a plane is an ellipse.

Hint. — Let the plane upon which the circle is projected intersect the plane of the circle in a diameter, and let the angle between the two planes be α. Then if a is the radius of the circle $b = a \cos \alpha$.

11. Given the circle $x^2 + y^2 = a^2$ with ordinates drawn from the x-axis to the circumference. Find the locus of a point which cuts off $1/n$th part of each ordinate.

12. Lines are drawn from the center of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ to its perimeter. Find the locus of the point which divides these lines in the ratio $1 : n$.

13. Two vertices of a triangle are $(\pm b, 0)$. Find the locus of the opposite vertex if the product of the tangents of the base angles is a^2/b^2.

14. Find the eccentricity of an ellipse when:

(*a*) the latus rectum is two-thirds the minor axis;
(*b*) the latus rectum is one-half the major axis.

15. Write the equation of an ellipse with center at the origin when:

(*a*) the curve passes through the points $(1, 4)$ and $(-6, 1)$.
(*b*) the minor axis is 6, and the distance from the focus to the vertex is 1.

16. The same as Problem 15 when:

(*a*) the curve passes through the points $(2, 4)$ and $(5, -3)$;
(*b*) the major axis is 16, the minor axis equals the distance between the foci.

17. A line of constant length moves so that its extremities A and B are on the x-axis and y-axis respectively. Find the locus of a point P on the line at a distance a from A and b from B.

18. A square is inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the coördinates of the vertices and the area of the square.

19. Show that if two ellipses have the same eccentricity, their major and minor axes are proportional.

20. The earth's orbit is an ellipse of eccentricity 0.01677, with the sun at the focus. The semi-major axis is 93 million miles. Find (*a*) the distance of the sun from the center; (*b*) the shortest distance from the sun to the earth.

21. Find the center, vertices, ends of the minor axis, latus rectum, and eccentricity of the ellipses:

(*a*) foci $(2, 2)$ and $(6, 2)$, sum of focal radii 8;
(*b*) foci $(3, -3)$ and $(3, 5)$, sum of focal radii 10.

22. Using the definition in § 62, derive the equation of each ellipse in Problem 21.

CHAPTER VII

THE HYPERBOLA

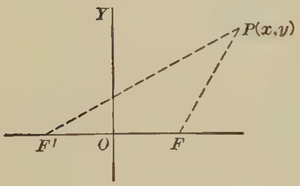
72. Definition. — *The hyperbola is the locus of a point such that the difference of its distances from two fixed points is a constant.*

As in the case of the ellipse, the two fixed points are called the *foci* and the line passing through them is called the *principal axis.* Let $2c$ be the distance between the foci, and let $2a$ be the difference of the distances of any point on the hyperbola from the foci.

73. The Equation of the Hyperbola. — Take the principal axis as the x-axis and the point midway between the foci F' and F as the origin. Then the coördinates of the foci are $(\pm c, 0)$ and any point $P(x, y)$ on the hyperbola must satisfy the relation

$$F'P - FP = 2a$$

or $$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = 2a.$$



If we rationalize this equation, as for the similar equation of the ellipse (§ 63), the result is

$$\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1.$$

It will be observed that this equation is identical in form with the equation of the ellipse (§ 63). But, in this case $F'P - FP < F'F$ or $2a < 2c$. Hence $c > a$, and $c^2 - a^2$ has a positive value for every hyperbola. Then we can write $c^2 - a^2 = b^2$ where b is always a real number, and the equation becomes,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{22}$$

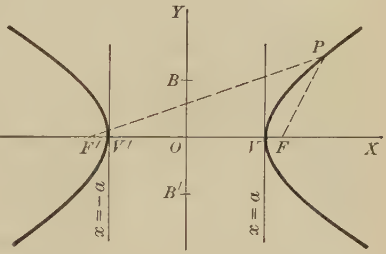
or

$$b^2x^2 - a^2y^2 = a^2b^2.$$

Exercise 1. Derive equation (22) as in § 63.

74. Discussion of the Hyperbola. — The intercepts on the x-axis are $\pm a$, the y-intercepts are imaginary. The curve is symmetrical with respect to both coördinate axes and with respect to the origin. Solving equation (22) for x, we have $x = \pm\frac{a}{b}\sqrt{y^2 + b^2}$, which shows that no value of y needs to be excluded.

Solving for y, we have $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$, which shows that y is imaginary for all values of x between $\pm a$, but if x is numerically greater than a, y is real and increases indefinitely as $|x|$ increases. Thus the hyperbola consists of two infinite branches, lying without the lines $x = \pm a$.



The points of the hyperbola V' and V on the principal axis are called the *vertices*. The segment $V'V$ of the principal axis is called the *transverse axis* and is of length $2a$. The segment $B'B$ of the y-axis, where $B'O = OB = b$, is called the *conjugate axis*. Since for the hyperbola, we have

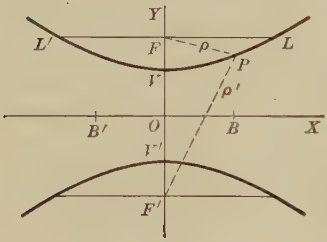
$$c^2 = a^2 + b^2 \tag{23}$$

c is greater than either a or b, while there is no restriction on the relative sizes of a and b. It is for this reason that the terms major and minor axis are not used.

75. Latus Rectum. — The chord $L'L$ through either focus perpendicular to the principal axis is called the latus rectum and it is seen (as in the case of the ellipse) that its length is $\frac{2b^2}{a}$.

Exercise 2. Show that the length of the latus rectum is $\frac{2b^2}{a}$.

76. Hyperbola with Foci on the Y-axis. — If we describe a hyperbola with the foci on the y-axis, with its transverse axis $2a$, and the distance between the foci $2c$, the effect on the original equation will be to interchange the variables x and y, and the equation will become



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \quad (22a)$$

or $b^2y^2 - a^2x^2 = a^2b^2$.

The vertices of this hyperbola are $(0, \pm a)$, the foci $(0, \pm c)$.

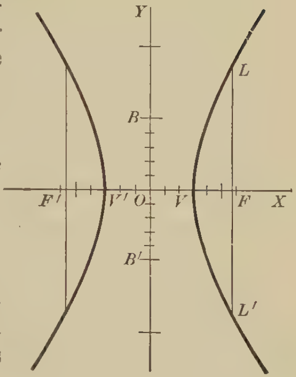
Exercise 3. Derive equation (22a) as in § 73.

77. Focal Radii. — The distances of any point on the hyperbola from the foci are called the *focal radii* and are designated by ρ and ρ'. From the definition of the hyperbola, we have

$$\rho' - \rho = \pm 2a \quad (24)$$

Hence *the difference of the distances of any point of the hyperbola from the foci is equal to the transverse axis.*

PROBLEMS



1. Find the semi-transverse and semi-conjugate axes, the latus rectum, and the foci, and draw the graphs of the following hyperbolas:

(*a*) $25x^2 - 9y^2 = 225$;

Solution. — Dividing by 225, we have the standard form

$$\frac{x^2}{9} - \frac{y^2}{25} = 1.$$

Hence $a = 3$, $b = 5$, $c = \sqrt{34}$. The latus rectum is $\frac{50}{3}$, and the foci are $(\pm\sqrt{34}, 0)$. As the vertices $(\pm 3, 0)$ give only two points, we lay off the semi-latus rectum $\frac{b^2}{a} = \frac{25}{3}$ from the foci, getting four more points.

(*b*) $\frac{x^2}{20} - \frac{y^2}{16} = 1$;

(*c*) $4y^2 - 9x^2 = 36$;

(*d*) $25x^2 - 16y^2 = 144$;

(*e*) $9y^2 - x^2 = 36$;

(*f*) $4x^2 - 25y^2 = 100$;

(*g*) $5x^2 - 3y^2 + 30 = 0$;

(*h*) $7x^2 - 5y^2 - 105 = 0$.

2. The same as Problem 1 for the following hyperbolas:

(*a*) $\frac{x^2}{16} - \frac{y^2}{9} = 1$;

(*b*) $\frac{y^2}{25} - \frac{x^2}{16} = 1$;

(*c*) $4x^2 - y^2 = 36$;

(*d*) $9x^2 - 25y^2 + 225 = 0$;

(*e*) $16x^2 - 36y^2 = 441$;

(*f*) $y^2 - 3x^2 = 27$;

(*g*) $25x^2 - 7y^2 - 350 = 0$;

(*h*) $x^2 - 9y^2 = -36$.

3. Find the equations of the hyperbolas having the center at the origin, first taking the transverse axis along the x-axis, and then taking the transverse axis along the y-axis, if

(*a*) $a = 4$, $b = \frac{5}{2}$;

(*b*) $b = 2\sqrt{5}$, $c = 6$;

(*c*) $a = 2\sqrt{6}$, $c = 7$;

(*d*) $a = 5$, latus rectum $\frac{32}{5}$;

(*e*) $b = 3\sqrt{3}$, latus rectum 9;

(*f*) $c = 7\frac{1}{2}$, latus rectum 16.

4. Find the equations of the hyperbolas as in Problem 3, if

(*a*) $a = 6$, $b = 3\sqrt{2}$;
(*b*) $c = 8$, $a = 4$;
(*c*) $b = 5$, $c = 7$;
(*d*) $a = 4\frac{1}{2}$, latus rectum 4;
(*e*) $b = 6$, latus rectum 15;
(*f*) $c = 4\sqrt{3}$, latus rectum 16.

5. Find the equation of the hyperbola with its center at the origin and the transverse axis of length $2a$ taken along the x-axis, if the vertex is mid-way between the center and the focus.

6. Find the equation of the hyperbola with its center at the origin and the conjugate axis of length $2b$ taken along the x-axis, and equal to one half the distance between the foci.

7. Find the equation of the hyperbola with its center at the origin and the transverse axis along the y-axis, if the curve passes through the points:

(*a*) $(0, 3)$ and $(4, -5)$;
(*b*) $(5, -10)$ and $(2, 5)$.

8. The same as Problem 7, except that the transverse axis is along the x-axis, and the curve passes through the points:

(*a*) $(3, -2)$ and $(\sqrt{3}, 1)$;
(*b*) $(8, 5)$ and $(-4, 1)$.

9. Find the intersections of the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{25} = 1 \text{ and the line } 25x - 12y + 45 = 0.$$

10. Find the intersections of the hyperbola

$$\frac{y^2}{16} - \frac{x^2}{36} = 1 \text{ and the line } 4x - 3y = 12.$$

11. Two vertices of a triangle are fixed at $(0, \pm c)$. Find the locus of the opposite vertex if the median to the fixed side is always the mean proportional between the other two sides.

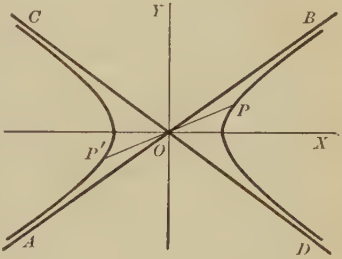
12. Two vertices of a triangle are at $(\pm a, 0)$. Find the locus of the opposite vertex if the product of the slopes of the variable sides is the constant $\frac{b^2}{a^2}$.

78. Eccentricity. — The shape of the hyperbola depends on the relative values of c and a. As in the case of the ellipse, the fraction $\frac{c}{a}$ is called the *eccentricity* of the hyperbola and is denoted by e, giving the same relation

$$\boldsymbol{c = ae}. \tag{21}$$

While the value of e for the ellipse is always less than 1, for the hyperbola it is always greater than 1.

79. Asymptotes. — The hyperbola has two asymptotes (§ 27) which intersect at the center of the curve. If a line is drawn through O intersecting the hyperbola at P, and P is made to move along the curve indefinitely, the line will turn about O and approach one of these two limiting lines, or asymptotes.



The equation of any line PP' through the center or origin is $y = mx$. The coördinates of its intersections with the hyperbola are given by solving this simultaneously with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Substituting, we have $\frac{x^2}{a^2} - \frac{m^2x^2}{b^2} = 1$, which gives

$$x = \pm \frac{ab}{\sqrt{b^2 - a^2m^2}}.$$

Now let P move along the curve indefinitely. Then x increases indefinitely and approaches the limit infinity. But the numerator ab is a constant, therefore the denominator decreases and approaches the limit 0. Hence PP' becomes an asymptote when

$$b^2 - a^2m^2 = 0 \qquad \text{or} \qquad m = \pm \frac{b}{a}.$$

Therefore the slopes of the asymptotes are $\pm\frac{b}{a}$ and their equations are

$$y = \pm\frac{b}{a}x, \tag{25}$$

or

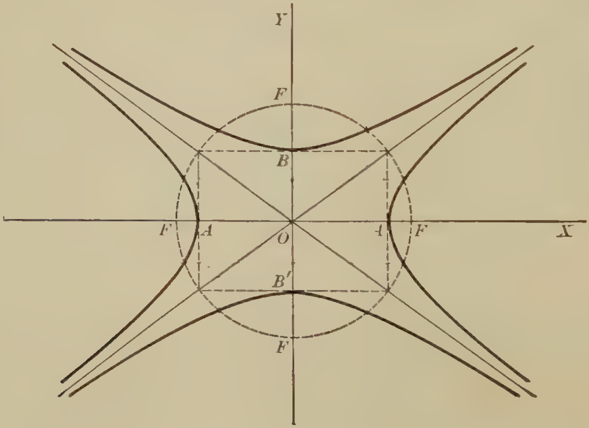
$$bx \pm ay = 0.$$

These equations may be combined by § 46 in the form

$$b^2x^2 - a^2y^2 = 0, \qquad \text{or} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \tag{25a}$$

If the vertices are on the y-axis, x and y are interchanged, giving $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 0$. In either case, *to find the equations of the asymptotes, write the equation of the hyperbola in the standard form, replace the* 1 *by* 0, *and factor.*

80. Conjugate Hyperbolas. — Since the relative sizes of a and b are immaterial, a hyperbola may be described having the segment 2 b for a transverse axis and the segment 2 a for a conjugate axis. The hyperbola with trans-



verse axis $A'A$ $(= 2a)$ along the x-axis and conjugate axis $B'B$ $(= 2b)$ along the y-axis has the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The hyperbola with transverse axis $B'B$ $(= 2b)$ along the y-axis and conjugate axis $A'A$ $(= 2a)$ along the x-axis has the equation $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. The two hyperbolas

$$(a)\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad (b)\ \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

are called *conjugate hyperbolas.* The transverse axis of each is the conjugate axis of the other.

Since $c^2 = a^2 + b^2$, the foci of both hyperbolas are at the same distance from the center. Those of (a) are $(\pm c, 0)$; those of (b) are $(0, \pm c)$. If we let e_1 and e_2 be the respective eccentricities, we have $e_1 = \frac{c}{a}$, $e_2 = \frac{c}{b}$, whence $e_1 : e_2 = b : a$. From the equations it is evident that conjugate hyperbolas have the same asymptotes.

This property is useful in sketching a pair of conjugate hyperbolas. Draw the rectangle of the given axes, that is, a rectangle having its sides of length $2a$ and $2b$ and symmetrical with respect to the axes, and draw the diagonals. As these have slopes $\pm\frac{b}{a}$, they are the asymptotes. The circle circumscribing the rectangle has the radius $\sqrt{a^2 + b^2} = c$, and therefore intersects the axes at the foci of the required hyperbolas. With the rectangle and asymptotes as guiding lines the curves can be sketched accurately.

Exercise 4. Show that for a pair of conjugate hyperbolas $e_1^2 + e_2^2 = e_1^2 e_2^2$.

81. Equilateral Hyperbolas. — When $b = a$ the equations of the conjugate hyperbolas reduce to the forms

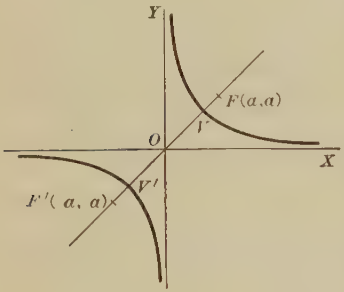
$$x^2 - y^2 = \pm a^2 \tag{26}$$

which may be written

$$(x - y)(x + y) = \pm a^2.$$

In this case $e_1 = e_2 = \sqrt{2}$, and the asymptotes are the perpendicular lines

$$x - y = 0 \quad \text{and} \quad x + y = 0.$$



Such hyperbolas are called *equilateral* or *rectangular* hyperbolas.

If the asymptotes of the equilateral hyperbolas are taken as the coördinate axes it can be shown that the equations take the important form

$$xy = \pm \frac{a^2}{2}. \tag{26a}$$

See page 138, Problems 3*c* and 4*c*.

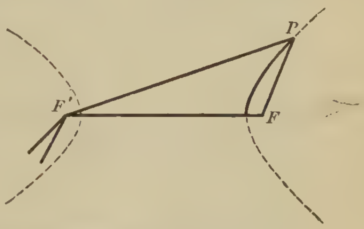
From this equation we see that the distances of any point on this curve from its asymptotes have a constant product. The last statement is also true for all hyperbolas.

Exercise 4. Using the definition of the hyperbola, show that its equation is $xy = \frac{a^2}{2}$ when the foci are taken at (a, a) and $(-a, -a)$, and the difference of the focal radii to any point is $2a$.

82. Construction of a Hyperbola. — A hyperbola may be plotted accurately by forming a table of values from the equation or by proceeding according to methods similar to those used for the ellipse (§ 70).

1. *Construction by points.* Lay off the distance $2a$ on coördinate paper. Now with any radius $\rho' > 2a$ and a focus as the center describe an arc. Then with a radius $\rho = \rho' - 2a$ and with the other focus as a center describe an arc. This will intersect the former arc at two points on the hyperbola. Thus any number of points can be readily located.

2. *Mechanical Construction.* Place thumb tacks in a drawing board at the foci F and F'. Let a string be tied to a pencil at P and looped about the tacks as in the figure. If the ends are drawn in together, P will describe a hyperbola since



$$(F'F + FP) - F'P$$

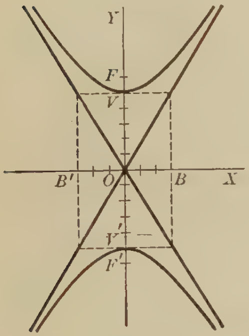
remains constant. For $F'F$ is constant; hence $F'P - FP$ is a constant.

83. Applications. — The principle of the hyperbola is used in rangefinding. Thus if the exact time of the report of a gun is recorded at each of two listening posts, the difference in time multiplied by the velocity of sound will give the difference of the distances of the gun from these two fixed points. Hence the gun is on the hyperbola whose foci are the listening posts, and whose transverse axis is the difference of the distances of the gun from them (§ 77). By taking two other listening posts another hyperbola can be found in the same way which passes through the position of the gun. Then its location is easily found at one of the intersections of these two curves.

If the elements of a right circular cone be produced through the vertex so as to form an equal inverted cone the intersections of any plane which meets both cones are the two branches of a hyperbola. (See Appendix).

PROBLEMS

1. Find the semi-transverse and semi-conjugate axes, the eccentricity, the coördinates of the foci, and the equations of the asymptotes of the following hyperbolas, and sketch each curve:



(*a*) $25x^2 - 9y^2 + 225 = 0$;

Solution. — Transposing and dividing by -225, we have

$$-\frac{x^2}{9} + \frac{y^2}{25} = 1.$$

Since the term in y is positive, this is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Hence $a = 5$, $b = 3$, $c = \sqrt{34}$, and $e = \frac{1}{5}\sqrt{34}$.

The foci are $(0, \pm\sqrt{34})$ and the asymptotes are the lines $y = \pm\frac{5}{3}x$. Drawing the asymptotes, the curve is readily sketched.

(*b*) $9x^2 - 4y^2 = 144$;
(*c*) $4x^2 - y^2 + 64 = 0$;
(*d*) $9x^2 - 25y^2 = 225$;
(*e*) $x^2 - y^2 + 25 = 0$;
(*f*) $9x^2 - 16y^2 + 144 = 0$;
(*g*) $x^2 - 4y^2 + 100 = 0$;
(*h*) $3x^2 - 5y^2 + 120 = 0$;
(*i*) $12x^2 - 5y^2 = 180$.

2. The same as Problem 1 for the following hyperbolas:

(*a*) $16x^2 - 9y^2 = 225$;
(*b*) $4x^2 - 25y^2 = -144$;
(*c*) $4x^2 - 5y^2 = 180$;
(*d*) $11x^2 - 5y^2 + 220 = 0$;
(*e*) $9x^2 - 16y^2 = 36$;
(*f*) $x^2 - y^2 + 16 = 0$;
(*g*) $4x^2 - 9y^2 + 144 = 0$;
(*h*) $16x^2 - 9y^2 = 144$.

3. Write the equations of the hyperbolas conjugate to those of Problem 1 and find the eccentricity and the coördinates of the foci of each.

4. Write the equations of the hyperbolas conjugate to those of Problem 2 and find the eccentricity and the coördinates of the foci of each.

5. Find the locus of a point which moves so that the difference of its distances from the points $(-5, 5)$ and $(5, -5)$ is 10.

6. Show that the distance of any point on the equilateral hyperbola $x^2 - y^2 = a^2$ from the center is a mean proportional between the focal radii.

7. Find the length of the perpendicular dropped from a focus of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ to its asymptote.

8. The ends of a line of variable length rest on two perpendicular lines, and the area of the triangle formed is a constant. Show that the locus of the mid-point of the variable line is an equilateral hyperbola.

9. Find the product of the lengths of the two perpendiculars from any point of the hyperbola $b^2x^2 - a^2y^2 + a^2b^2 = 0$ to its asymptotes.

10. Find the coördinates of the vertices, the eccentricity, and the equations of the asymptotes of the following hyperbolas, and sketch each curve.

(*a*) Foci $(-2, 2)$ and $(8, 2)$, difference of focal radii to any point on the curve is 6.

(*b*) Foci $(4, 3)$ and $(4, -9)$, length of semi conjugate axis is $2\sqrt{5}$.

11. Find the foci, vertices, and eccentricity of the hyperbolas conjugate to those in Problem 10.

12. Derive the equations of the hyperbolas in Problem 10.

Hint. — Solve as a locus problem using definition § 72.

13. Find the equations of the following hyperbolas:

(*a*) foci $(\pm 6, 0)$, $e = 2$; (*c*) vertices $(\pm 2\sqrt{5}, 0)$, $e = \sqrt{2}$;
(*b*) vertices $(0, \pm 4)$, $e = \frac{3}{2}$; (*d*) foci $(0, \pm 5)$, $e = \frac{1}{2}\sqrt{5}$.

14. Find the equation of the hyperbolas conjugate to those in Problem 13.

15. An ellipse and a hyperbola each have foci at $(\pm c, 0)$ and the eccentricities are $\frac{1}{2}$ and 2 respectively. Find the coördinates of their intersection.

16. If an ellipse and a hyperbola have the same foci and same latera recta show that their eccentricities are reciprocals.

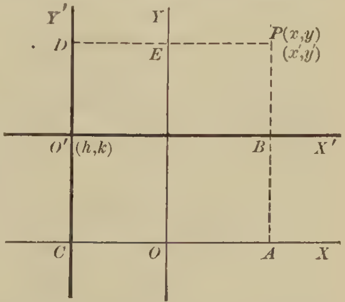
CHAPTER VIII

TRANSFORMATION OF COÖRDINATES AND SIMPLIFICATION OF EQUATIONS

84. Change of Axes. — The position of the axes of coördinates to which a given locus is referred is arbitrary. It may be changed at will and the equation of the locus altered to correspond by substituting for the former coördinates their values in terms of coördinates measured from the new axes.

The advantage to be secured by a change of axes is usually a simplification of the equation, and this is best secured by choosing a position of symmetry when possible.

When the new axes are drawn parallel to the old, the transformation of the equation is called a *transformation by translation*. When they are drawn through the same origin oblique to the axes but still perpendicular to each other, the transformation is called a *transformation by rotation*.



85. Formulas of Translation. — In the figure let the original set of axes be OX and OY; the new axes $O'X'$ and $O'Y'$ with the origin at O', which has the coördinates (h, k) with reference to OX and OY. Let P be any point in the plane which has coördinates (x, y) with reference to the old axes, and (x', y') with reference to the new ones. Then

$$x = EP, \qquad x' = DP, \qquad h = OC = ED,$$
$$y = AP, \qquad y' = BP, \qquad k = CO' = AB.$$

From the figure

$$EP = DP - DE = DP + ED \quad \text{and} \quad AP = BP + AB.$$

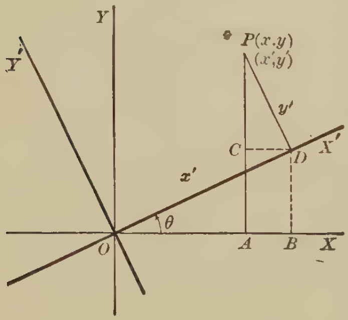
Hence
$$\boldsymbol{x = x' + h,}$$
and
$$\boldsymbol{y = y' + k.} \tag{27}$$

These are the *formulas of translation.*

By using these substitutions the equation of a given locus is transformed into a new equation in x' and y', which is the equation of the same locus referred to coördinate axes drawn through the point (h, k) parallel to the old axes.

86. Formulas of Rotation. — Let the original axes be OX and OY, the new axes OX' and OY', and the angle of rotation θ. P is any point in the plane with coördinates (x, y) and (x', y') with reference to the old and new axes respectively. Then

$$x = OA, \qquad x' = OD,$$
$$y = AP, \qquad y' = DP.$$



Since the sides of angle CPD are perpendicular to those of θ, $CPD = \theta$. Hence we have

$$CD = y' \sin\theta, \qquad OB = x' \cos\theta,$$
$$CP = y' \cos\theta, \qquad BD = x' \sin\theta.$$

But $x = OA = OB - CD$ and $y = AP = BD + CP$. Substituting, we get

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta, \\ y &= x' \sin\theta + y' \cos\theta. \end{aligned} \qquad (28)$$

which are the *formulas of rotation.*

These substitutions transform a given equation into a new equation with variables x' and y', in which the coördinate axes are drawn through the old origin but inclined at an angle θ to the old axes.

PROBLEMS

General Directions. — Usually the equation obtained by transformation is a known form and its locus can be identified. When this is not the case, the table of values should be computed from the new equation and the curve plotted on the new axes. Frequently the intercepts with respect to both old and new axes, together with considerations of symmetry, make an extended table of values unnecessary.

1. Translate the origin to the point indicated and transform the equation to correspond. Draw the curve and both sets of axes.

(*a*) $4x^2 - y^2 - 8x + 4y + 4 = 0, (1, 2)$;
(*b*) $2x^2 + y - 5 = 0, (0, 5)$;
(*c*) $y^2 - 8x = -24, (3, 0)$;
(*d*) $x^2 + y^2 = 6x - 4y, (3, -2)$;
(*e*) $y = x^3 + 6x^2 + 8x + 3, (-2, 3)$;
(*f*) $xy - kx - hy = c^2 - hk, (h, k)$;

2. The same as Problem 1 for the following:

(*a*) $y^2 + 4x + 4y = 4, (2, -2)$;
(*b*) $9x^2 + 4y^2 + 54x - 32y + 1 = 0, (-3, 4)$;
(*c*) $x^2 - 2mx - ky + kn + m^2 = 0, (m, n)$;
(*d*) $16x^2 - 9y^2 - 64x - 18y - 66 = 0, (2, -1)$;
(*e*) $2xy - 8x + 5y - 20 = 0, (-2\frac{1}{2}, 4)$;
(*f*) $y = b + (x - a)^3, (a, b)$.

3. Rotate the axes through the angle indicated and transform the equation to correspond. Draw the curve and both sets of axes. Identify known loci.

(*a*) $xy = 8,\ \theta = \frac{\pi}{4}$;

(*b*) $x^2 - y^2 = 16,\ \theta = \frac{\pi}{4}$;

(*c*) $xy = \pm\frac{a^2}{2},\ \theta = \frac{\pi}{4}$;

(*d*) $x^2 + y^2 - 4xy + 9 = 0,\ \theta = \frac{\pi}{4}$;

(*e*) $17x^2 - 312xy + 108y^2 = 900,\ \theta = \sin^{-1}\frac{3}{5}$;

(*f*) $x^2(a + x) = y^2(a - x),\ \theta = \frac{\pi}{4}$.

4. The same as Problem 3 for the following:

(*a*) $xy = -\frac{25}{2},\ \theta = \frac{\pi}{4}$;

(*b*) $x^2 - y^2 + 25 = 0,\ \theta = \frac{\pi}{4}$;

(*c*) $x^2 - y^2 = \pm a^2,\ \theta = \frac{\pi}{4}$;

(*d*) $x^2 + 4xy + y^2 = 16,\ \theta = \frac{\pi}{4}$;

(*e*) $29x^2 - 24xy + 36y^2 = 180,\ \theta = \sin^{-1}\frac{3}{5}$;

(*f*) $(x^2 + y^2)^2 = 2a^2(x^2 - y^2),\ \theta = -\frac{\pi}{4}$.

5. Find the coördinates of the points $P_1(-3, 4)$ and $P_2(a, b)$ when the origin is translated to

(*a*) $(6, 5)$; (*b*) $(-4, 2)$; (*c*) $(-3, -4)$; (*d*) $(h, -k)$.

6. Find the coördinates of the points $P_1(4, -5)$ and $P_2(-c, d)$ when the origin is translated to

(*a*) $(4, 3)$; (*b*) $(-2, 6)$; (*c*) $(-3, -6)$; (*d*) $(-h, -k)$.

7. Find the coördinates of

(*a*) $(3, 5)$; (*b*) $(-2, 2)$; (*c*) (h, k),

when the axes are rotated −45°; 90°.

8. Find the coördinates of

(*a*) $(4, -2)$; (*b*) $(-6, -6)$; (*c*) $(-h, k)$,

when the axes are rotated 45°; −90°.

9. The coördinates of a point P are (x, y); after a rotation of the axes through an angle θ these become (x', y'). Derive formulas expressing x' and y' in terms of x and y.

10. Prove that $x^2 + y^2 = r^2$ is unchanged by rotating the axes through any angle.

87. Test for Axes of Symmetry. — To find axes of symmetry parallel to the coördinate axes, proceed as follows: Solve the equation for y in terms of x. If the solution is of the form $y = k \pm f(x)$, the line $y = k$ is an axis of symmetry. This is due to the fact that each value of x, as $x = a$, gives two values of y, and thus two points, one $f(a)$ above the line $y = k$, the other $f(a)$ below. Similarly, if the solution for x is of the form $x = h \pm g(y)$, the line $x = h$ is an axis of symmetry. Since the lines $x = h$ and $y = k$ are perpendicular to each other, a curve symmetrical with respect to both of these lines has their intersection (h, k) as a center of symmetry. (See § 25.)

88. Simplification of Equations by Translation. — If the curve has two axes of symmetry, they can be found by the method just given and the translation made by formula. When this is not true, the second method illustrated in the following examples must be used.

Example 1. — Simplify $x^2 + 4y^2 - 2x - 24y + 21 = 0$.

Solution. — First Method. Solving for x and y, we have

$$x = 1 \pm f(y)^*$$

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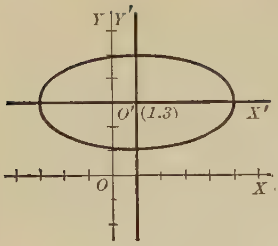
* In solving for x, the terms not involving x may be disregarded, as we are interested only in showing that the solution is of the form

$$x = h \pm f(y).$$



and $$y = 3 \pm g(x).$$

Hence the axes of symmetry are

$$x = 1 \quad \text{and} \quad y = 3$$

and the center of symmetry is (1, 3).

Substitute

$$x = x' + 1 \text{ and } y = y' + 3$$

in the original equation, and we have

$$(x' + 1)^2 + 4(y' + 3)^2 - 2(x' + 1) - 24(y' + 3) + 21 = 0.$$

$$x'^2 + 4y'^2 = 16.$$

$$\frac{x'^2}{16} + \frac{y'^2}{4} = 1.$$

Thus the curve is an ellipse with its center at (1, 3) and semi-axes 4 and 2.

Second Method. In the given equation substitute $x = x' + h$ and $y = y' + k$. It becomes, after collecting coefficients,

$$x'^2 + (2h - 2)x' + 4y'^2 + (8k - 24)y' + (h^2 + 4k^2 - 2h - 24k + 21) = 0.$$

If the curve is symmetrical with respect to the new axes, there can be no terms of the first degree in the new equation. This is the case if

$$2h - 2 = 0 \quad \text{and} \quad 8k - 24 = 0, \quad \text{or} \quad h = 1, \quad k = 3.$$

Hence the center of symmetry is (1, 3) and the transformed equation becomes $x'^2 + 4y'^2 - 16 = 0$ as before, on substituting $h = 1$, $k = 3$.

EXAMPLE 2. — Simplify $y^2 + 8x + 4y - 20 = 0$.

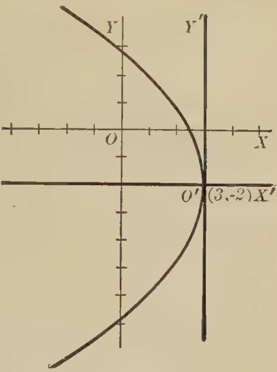
Solution. — If the equation is solved for x no axis of symmetry is revealed, and so we must use the second method. Substitute $x = x' + h$, and $y = y' + k$. Collecting coefficients, we have

$$y'^2 + 8x' + (2k + 4)y' + (k^2 + 8h + 4k - 20) = 0.$$

Evidently the terms in y'^2 and x' cannot be eliminated. But if $2k + 4 = 0$ and $k^2 + 8h + 4k - 20 = 0$, the term in y' and the constant term vanish. These equations give $k = -2$, $h = 3$. The new equation then is

$$y'^2 = -8x',$$

which is of the form $y^2 = -2px$. The curve is a parabola which has its vertex at $(3, -2)$ and its axis $y = -2$.



PROBLEMS

1. Find the center of symmetry by solving for x and y, and move the origin to this point. Draw the curve and both sets of axes.

(a) $9x^2 - 36x + 4y^2 + 16y + 16 = 0$;
(b) $16x^2 - 64x - 25y^2 - 336 = 0$;
(c) $4x^2 - y^2 + 32x + 4y + 60 = 0$;
(d) $4x^2 + 5y^2 + 4x - 40y + 1 = 0$;
(e) $x^2 + y^2 - 10x - 24y = 0$.

2. The same as Problem 1 for the following:

(a) $25x^2 - 150x + 9y^2 + 36y + 36 = 0$;
(b) $16x^2 - 4y^2 + 16x - 16y - 93 = 0$;
(c) $x^2 + y^2 - 18x + 6y + 54 = 0$;
(d) $x^2 + x + 3y^2 - 9y + 4 = 0$;
(e) $6x^2 - 4y^2 - 12x + 16y + 34 = 0$.

3. Simplify the following equations by the second method. Draw the curve and both sets of axes.

(a) $4x^2 + y^2 + 16x - 6y = 39$;
(b) $16x^2 - y^2 + 32x - 6y = 74$;
(c) $4x^2 - 4x - 32y - 15 = 0$;
(d) $y^2 + 6y - 3x + 6 = 0$;
(e) $y = x^3 - 4x^2 + 3x$.

4. The same as Problem 3 for the following:

(*a*) $2x^2 - 6x + 5y + 17 = 0$;
(*b*) $6x + y^2 + 8y = 0$;
(*c*) $3x^2 - 18x + 4y^2 - 8y - 5 = 0$;
(*d*) $4x^2 + 16x + y^2 - 12y + 16 = 0$;
(*e*) $y = x^3 + 9x^2 + 27x + 27$.

5. Find the equation of each of the following parabolas and translate the origin to the vertex:

(*a*) focus (4, 0), directrix $x = 0$;
(*b*) focus (1, 2), directrix $y = -4$;
(*c*) focus (2, −3), vertex (5, −3);
(*d*) focus (−2, 3), vertex (−2, 8).

6. Find the equation of each of the following parabolas and translate the origin to the vertex:

(*a*) focus (3, −4), directrix $y = 0$;
(*b*) focus (−2, 3), directrix $x = 1$;
(*c*) focus (0, 0), vertex (0, 2);
(*d*) focus (−2, 3), vertex (3, 3).

7. Find the equation of each of the following ellipses and translate the origin to the center of symmetry:

(*a*) foci (−1, 4) and (7, 4), sum of focal radii 10;
(*b*) foci (2, 6) and (2, 0), sum of focal radii 8.

8. Find the equation of each of the following hyperbolas and translate the origin to the center of symmetry:

(*a*) foci (−1, −2) and (9, −2), difference of focal radii 6;
(*b*) foci (2, 3) and (2, −9), difference of focal radii 8.

In each of the following locus problems simplify the equation by a translation of the axes.

9. Find the locus of the middle points of chords of the ellipse $4x^2 + y^2 = 144$ drawn from one end of (*a*) the major axis; (*b*) the minor axis.

10. From one focus of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, focal radii are drawn and bisected. Show that the locus of the points of bisection is an ellipse, and find its center and foci.

11. In each of the following find the equation of the locus and simplify by a translation of the axes:

(*a*) Page 32, Problem 8*a*;	(*d*) Page 109, Problem 31*c*;
(*b*) Page 32, Problem 10*a*;	(*e*) Page 121, Problem 22*b*;
(*c*) Page 32, Problem 12*a*;	(*f*) Page 133, Problem 12*a*.

12. Find the locus of the centers of all circles tangent to the circle $x^2 + y^2 = a^2$ and the line $y = b$, and simplify the equation by a translation of the axes.

89. Discussion of the Equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$. — This represents the general form of a quadratic in x and y with the xy term lacking. The only restriction on the coefficients is that A and C cannot both be zero. We desire to find under what conditions the locus is a parabola, an ellipse, or a hyperbola.

CASE 1. *When either A or C is zero.* Let $A = 0$. Here the equation has the form

$$Cy^2 + Dx + Ey + F = 0.$$

If $D \neq 0$, by a proper translation we can remove* the y term and the constant, and obtain an equation of the form $Cy'^2 + Dx' = 0$, of which the locus is a parabola.

If $D = 0$, the y term can be removed by translation, and we get $Cy'^2 + F' = 0$, of which the locus is imaginary if F' and C have the same sign; the new x-axis if $F' = 0$; and a pair of lines parallel to the x-axis if F' and C have unlike signs.

A similar discussion holds when $C = 0$.

---

* By actual substitution of $x = x' + h$ and $y = y' + k$, it can be shown that (*a*) translation does not affect the coefficients of the highest powers of the variables; (*b*) that, if neither A nor C is zero, the first-degree terms can be removed; (*c*) that, if one variable appears only to the first degree, the first power of the other variable and the constant term can be removed; (*d*) that, if one variable is missing, the first power of the other can be removed.

Case 2. *When A and C are of like sign.* Removing the first-degree terms by translation, we have

$$Ax'^2 + Cy'^2 = F'.$$

This is evidently an ellipse if F' has the same sign as A and C; a point if $F' = 0$; and an imaginary locus if F' has the opposite sign to that of A and C.

Case 3. *When A and C are of unlike sign.* Removing the first-degree terms, we have

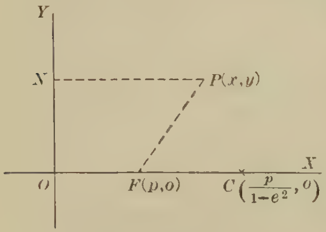
$$Ax'^2 + Cy'^2 = F'.$$

This is evidently a hyperbola unless $F' = 0$, in which case the locus is a pair of lines intersecting at the new origin.

90. Conics. — Any plane section of a right circular cone is an ellipse, parabola, hyperbola, or a limiting form of one of these curves. Hence they are called *conic sections* or *conics.**

Although these curves were defined separately in the preceding chapters, they are all included in the following general definition:

A conic is the locus of a point such that its distance from a fixed point has a constant ratio to its distance from a fixed line.



The fixed line is called a directrix. Let p be the distance of the fixed point F from the directrix and denote the constant ratio by e. Take the directrix as the y-axis and let F be on the x-axis. Then any point P on the conic must satisfy the relation

$$FP = e \cdot NP$$

---

* A proof of this interesting theorem is given in the Appendix.

or
$$\sqrt{(x-p)^2+y^2}=ex.$$

Squaring and collecting terms we have

$$(1-e^2)\,x^2-2\,px+y^2+p^2=0. \tag{29}$$

If $e = 1$, the coefficient of x^2 is zero and the locus is a parabola. See Case 1, § 89.

If $e < 1$, the coefficients of x^2 and y^2 have like signs and the locus is an ellipse. See Case 2, § 89.

If $e > 1$, the coefficients of x^2 and y^2 have unlike signs and the locus is a hyperbola. See Case 3, § 89.

By completing the square, (29) becomes

$$\left(x-\frac{p}{1-e^2}\right)^2+\frac{y^2}{1-e^2}=\left(\frac{ep}{1-e^2}\right)^2.$$

Hence, by the first method of § 88, the point $\left(\frac{p}{1-e^2}, 0\right)$ is a center of symmetry, and if the origin is translated to this point the new equation is

$$x^2+\frac{y^2}{1-e^2}=\left(\frac{ep}{1-e^2}\right)^2 \quad \text{or} \quad \frac{x^2}{\left(\frac{ep}{1-e^2}\right)^2}+\frac{y^2}{\frac{(ep)^2}{1-e^2}}=1.$$

If $e < 1$ this equation is of the type $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and therefore $a=\frac{ep}{1-e^2}$, and $b=\frac{ep}{\sqrt{1-e^2}}=a\sqrt{1-e^2}$.

Since for the ellipse $c^2 = a^2 - b^2$ we have on substituting

$$c^2=a^2e^2 \quad \text{or} \quad c=ae \quad \text{and} \quad e=\frac{c}{a}\cdot$$

Hence the constant ratio e is the eccentricity of the ellipse. Also the distance from F to the center C is seen to be

$$OC-OF=\frac{p}{1-e^2}-p=\frac{e^2p}{1-e^2}=ae=c.$$

Hence the fixed point F is a focus of the ellipse and, since the ellipse is symmetrical with respect to its center, there must be a directrix corresponding to each focus.

The student should show that these same results are true for the hyperbola by taking $e > 1$ and using the relation $c^2 = a^2 + b^2$.

Exercise 1. Show that for either the ellipse or the hyperbola the distance between the center and a directrix is $\frac{a}{e}$.

Exercise 2. Using the definition of the conic find its equation when the focus is at $(c, 0)$ and the equation of the directrix is $x = \frac{a}{e}$.

91. General Statement. — The preceding analysis may be summed up as follows:

A quadratic equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

always defines a conic or one of its limiting forms, the coefficients being any real numbers, zero included.

If either A or C is zero, the conic is a parabola or in special cases two parallel straight lines distinct, coincident, or imaginary.

If A and C have the same sign, the conic is an ellipse, or in special cases a circle, point, or imaginary ellipse.

If A and C have unlike signs, the conic is a hyperbola, or in special cases two intersecting straight lines.

92. Generalized Standard Equations of the Conics. — The equations derived in Chapters V, VI, and VII we have seen to be of a special character, since they are applicable only to the case that the vertex or center is at the origin and the principal axis is the x- or y-axis. We are now in a position to generalize these equations and obtain standard forms where the only restriction is that the principal axis is parallel to one of the coördinate axes.

Consider an ellipse with the center at (h, k), semi-axes a and b, and transverse axis $y = k$. It is required to find its equation.

If the origin is translated to (h, k), we know from Chapter VI that the transformed equation will be

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

The problem then is the reverse of that considered in § 88, viz., the transformed equation is given and the equation with reference to the old axes is to be found. Hence the substitutions are those of § 85 reversed, that is, $x' = x - h$, and $y' = y - k$. These give

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

as the required equation.

In the same way we discuss the other cases and obtain the following equations:

Parabola —

Vertex (h, k), axis $y = k$: $(y - k)^2 = 2p(x - h)$; (30)

" " " $x = h$: $(x - h)^2 = 2p(y - k)$; (30*a*)

Ellipse —

Center (h, k), axis $y = k$: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; (31)

" " " $x = h$: $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$; (31*a*)

Hyperbola —

Center (h, k), axis $y = k$: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$; (32)

" " " $x = h$: $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. (32*a*)

These forms are easy to remember if one bears in mind that they are merely generalizations of the simple forms $y^2 = 2\,px$, $x^2 = 2\,py$, etc. To reduce an equation to one of these forms, it is necessary only to complete the squares.

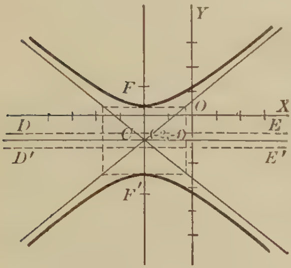
Example 1. — Write the equation of the ellipse which has the center (0, 1), one focus (−4, 1), and minor axis 6.

Solution. — Here the principal axis is $y = 1$. From the given data $b = 3$ and $c = 4$, whence $a = 5$. Therefore the equation is

$$\frac{(x-0)^2}{25} + \frac{(y-1)^2}{9} = 1.$$

Example 2. — Simplify the equation $2\,x^2 - 3\,y^2 + 8\,x - 6\,y + 11 = 0$. Find the center, semi-axes, vertices, and foci, and draw the curve.

Solution. — Completing squares, we have



$$2\,(x+2)^2 - 3\,(y+1)^2 = -6$$

or

$$-\frac{(x+2)^2}{3} + \frac{(y+1)^2}{2} = 1.$$

The center is $(-2, -1)$ and since the term in $(y+1)^2$ is positive, the transverse axis is $x = -2$. Hence $a = \sqrt{2}$, $b = \sqrt{3}$, and $c = \sqrt{5}$. The vertices are

$$(-2, -1 \pm \sqrt{2})$$

and the foci

$$(-2, -1 \pm \sqrt{5}).$$

PROBLEMS

1. Write the equations of the following ellipses in the standard form:

(*a*) vertices $(\pm 5, 3)$, focus $(3, 3)$;
(*b*) vertices $(1, 7)$, $(1, -3)$, focus $(1, 0)$;
(*c*) vertices $(-4, -2)$, $(8, -2)$, $e = \frac{2}{3}$;
(*d*) foci $(-3, 7)$, $(-3, 1)$, $e = \frac{3}{5}$.

2. Write the equations of the following ellipses in the standard form, the major axis being first parallel to the x-axis, and second parallel to the y-axis:

(*a*) $a = 5$, $b = 3$, center $(2, 3)$;
(*b*) $a = 4$, $b = 2$, center $(-1, 2)$,
(*c*) $b = 4$, $c = \sqrt{20}$, center $(-4, 6)$;
(*d*) $b = 5$, $c = 7$, center $(-3, -1)$.

3. Write the equations of the following hyperbolas in the standard form:

(*a*) vertices $(-2, -1)$, $(6, -1)$, focus $(8, -1)$;
(*b*) foci $(3, 8)$, $(3, 0)$, vertex $(3, 2)$;
(*c*) foci $(\pm 6, 2)$, $e = \frac{3}{2}$;
(*d*) vertices $(-4, 7)$, $(-4, -1)$, $e = 2$.

4. Write the equations of the hyperbolas satisfying the data of Problem 2.

5. Write the equations of the hyperbolas conjugate to those in Problem 4.

6. Write the equations of the asymptotes to the hyperbolas in Problem 3.

7. Write by inspection the equations of the parabolas having:

(*a*) focus $(0, 0)$, vertex $(3, 0)$;
(*b*) vertex $(2, -4)$, directrix $x + 1 = 0$;
(*c*) focus $(-5, 1)$, directrix $y + 6 = 0$;
(*d*) focus (a, b), vertex $(a, 3b)$;
(*e*) vertex (a, b), directrix $x = 2a$.

8. Write by inspection the equations of the parabolas having:

(*a*) vertex $(2, 5)$, focus $(2, 1)$;
(*b*) focus $(0, 5)$, directrix $x = 6$;
(*c*) vertex $(-1, 7)$, directrix $y = 3$;
(*d*) focus (c, d), vertex $(-c, d)$;
(*e*) focus $(h + a, k)$, directrix $x = h - a$.

9. Simplify the following equations by completing the squares and draw their loci:

(*a*) $16x^2 + 9y^2 - 64x - 54y + 1 = 0$;
(*b*) $x^2 + 2x - 7y + 29 = 0$;
(*c*) $2x^2 - 3y^2 + 12x - 12y - 42 = 0$;
(*d*) $2y^2 - 9x + 12y + 27 = 0$;
(*e*) $3x^2 + 8y^2 + 12x + 8y - 82 = 0$;
(*f*) $4x^2 - 9y^2 - 4x - 12y + 105 = 0$.

10. Simplify the following equations by completing the squares and draw their loci:

(*a*) $2x^2 + 7y^2 + 4x - 42y - 5 = 0$;
(*b*) $2x^2 - 6x + 9y + 9 = 0$;
(*c*) $6x^2 - 4y^2 + 24x + 4y - 73 = 0$;
(*d*) $3y^2 + 15x + 8y + 2 = 0$;
(*e*) $9x^2 + 4y^2 - 30x + 28y - 34 = 0$;
(*f*) $25x^2 - 9y^2 - 60y = 0$.

11. Show that the locus of $y = a + bx + cx^2$, where a, b, and c are constants and $c \neq 0$, is a parabola whose axis is parallel to the y-axis and which extends upwards from the vertex if $c > 0$, and downwards if $c < 0$.

12. Find the vertex and focus and draw the graph of each of the following parabolas:

(*a*) $4y = 8x + x^2$;
(*b*) $y = 3 - 4x + x^2$;
(*c*) $12y = 12x - x^2$;
(*d*) $y = x^2 + 5x + 4$;
(*e*) $4x = 8 + 8y - y^2$;
(*f*) $x = y - y^2$.

13. The following numbers refer to curves on page 41. Reduce each equation to one of the standard forms of § 92 and make a sketch.

(*a*) 4;
(*b*) 5;
(*c*) 6;
(*d*) 10;
(*e*) 13;
(*f*) 14;
(*g*) 15;
(*h*) 16;
(*i*) 23.

14. For which of the equations on page 41 are the loci (*a*) parabolas; (*b*) ellipses; (*c*) hyperbolas?

93. Simplification of Equations by Rotation. — The process of rotation of the axes effects the removal of the xy term from the second degree equation if the proper angle θ is chosen. Let us substitute in the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the rotation formulas

$$x = x' \cos \theta - y' \sin \theta,$$
$$y = x' \sin \theta + y' \cos \theta.$$

The coefficient of the $x'y'$ term will be

$$-2\,A \sin \theta \cos \theta + B\,(\cos^2 \theta - \sin^2 \theta) + 2\,C \sin \theta \cos \theta,$$

or
$$-A \sin 2\,\theta + B \cos 2\,\theta + C \sin 2\,\theta.$$

(The student should make the substitution in full for all terms of the general equation.)

The term in $x'y'$ will vanish if

$$(C - A) \sin 2\,\theta + B \cos 2\,\theta = 0,$$

or if

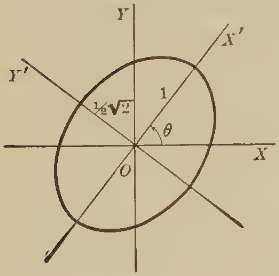
$$\tan 2\,\theta = \frac{B}{A - C}. \tag{33}$$

Thus we have the theorem:

For any second degree equation there is an angle of value less than 90° *such that the substitutions for rotation through this angle will transform the equation into one containing no xy term.*

For the equation $\tan 2\,\theta = \dfrac{B}{A - C}$ gives values of $\tan 2\,\theta$, positive or negative, from 0 when $B = 0$ to infinity when $A = C$. Hence for all values of A, B, and C, $2\,\theta$ will have some value between 0 and 180°.

Example. — Simplify $41x^2 - 24xy + 34y^2 = 25$.



Solution. — Here $A = 41$, $B = -24$, and $C = 34$.

Therefore $\tan 2\theta = \frac{-24}{7}$.

This gives at once

$$\cos 2\theta = \frac{-7}{25}.$$

Substituting in the half-angle formulas of trigonometry,

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \frac{4}{5},$$

and

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \frac{3}{5}.$$

Hence $x = \frac{3}{5}x' - \frac{4}{5}y'$ and $y = \frac{4}{5}x' + \frac{3}{5}y'$.

Substituting in the above equation and reducing, we have

$$x'^2 + 2y'^2 = 1,$$

an ellipse of semi-axes 1 and $\frac{\sqrt{2}}{2}$.

94. The General Equation of the Second Degree. — We have seen that the equations of the conics are all special cases of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

We are now ready to prove the converse theorem; namely,

Every equation of the second degree in two variables defines a conic or one of the limiting forms of the conic.

It has been shown in § 89 that this theorem is true for every form of the second-degree equation in which the product xy does not appear. Also we have just seen in the previous section that by a proper rotation of the axes the xy term can be made to disappear from the equation. Thus the proof of the theorem is complete.

95. The Characteristic. — The quantity $B^2 - 4\,AC$ is called the characteristic of the general second-degree equation, and is denoted by Δ. We now prove the theorem:

The characteristic of a general equation of the second degree is unaltered by a rotation of the axes through any angle θ.

In the general equation substitute the values given in the rotation formulas. The first three terms become

$$\begin{array}{r|l r|l r|l}
A\cos^2\theta & x'^2 & -2\,A\sin\theta\cos\theta & x'y' & +A\sin^2\theta & y'^2 \\
+B\sin\theta\cos\theta & & -B\sin^2\theta & & -B\sin\theta\cos\theta & \\
 & & +B\cos^2\theta & & & \\
+C\sin^2\theta & & +2\,C\sin\theta\cos\theta & & +C\cos^2\theta &
\end{array}$$

Calling the coefficients of the new equation A', B', C', etc., we have:

$$A' = A\cos^2\theta + C\sin^2\theta + B\sin\theta\cos\theta,$$
$$C' = A\sin^2\theta + C\cos^2\theta - B\sin\theta\cos\theta,$$
$$B' = 2\,C\sin\theta\cos\theta - 2\,A\sin\theta\cos\theta + B\cos^2\theta - B\sin^2\theta.$$

The characteristic of the new equation, $B'^2 - 4\,A'C'$, becomes on multiplying and collecting the terms:

$$\begin{aligned}
&B^2\cos^4\theta + B^2\sin^4\theta - 8\,AC\sin^2\theta\cos^2\theta - 4\,AC\sin^4\theta - \\
&\qquad\qquad 4\,AC\cos^4\theta + 2\,B^2\sin^2\theta\cos^2\theta \\
&= (B^2 - 4\,AC)\cos^4\theta + (B^2 - 4\,AC)\sin^4\theta + 2\,(B^2 - \\
&\qquad\qquad 4\,AC)\sin^2\theta\cos^2\theta \\
&= (B^2 - 4\,AC)\,(\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta) \\
&= (B^2 - 4\,AC)\,(\cos^2\theta + \sin^2\theta)^2 \\
&= B^2 - 4\,AC.
\end{aligned}$$

It is easy to see that Δ is also unchanged by a translation of the axes. For this reason it is called an *invariant* of the equation.

Exercise 3. Prove that $A + C$ is not changed by the substitutions for rotation or translation of the axes.

96. Test for distinguishing the Conics. — If in the previous section, the angle θ were chosen so that the xy term vanished, the new equation would be

$$A'x'^2 + C'y'^2 + Dx' + Ey' + F = 0$$

and $\quad \Delta = B^2 - 4\,AC = B'^2 - 4\,A'C' = -4\,A'C',$

since $B' = 0$.

From this relation and § 91 we see that

if $\Delta < 0$, *A' and C' are of like sign, and the conic is an ellipse,**

if $\Delta = 0$, *A' or C' is zero, and the conic is a parabola,**

if $\Delta > 0$, *A' and C' are of unlike sign, and the conic is a hyperbola.**

97. Suggestions for simplifying the Equation of a Conic. — If the conic is an ellipse or hyperbola (that is, if $\Delta \neq 0$), determine the coördinates (h, k) of the center of symmetry and remove the first-degree terms by translation. Then rotate the axes as in the example of § 93.

If the conic is a parabola ($\Delta = 0$), the substitutions for rotation should be made first, and then equations of condition involving h and k can be formed.

A convenient check on the accuracy of the rotation substitutions is the fact that $A + C$ is unchanged until the new equation is simplified. (See Exercise 3, § 95.)

PROBLEMS

Simplify the following equations. In each case draw the conic and the three sets of coördinate axes.†

1. $73\,x^2 + 72\,xy + 52\,y^2 - 218\,x - 176\,y + 97 = 0.$

2. $66\,x^2 + 24\,xy + 59\,y^2 + 396\,x + 72\,y + 444 = 0.$

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* Or one of the limiting forms discussed in §§ 89, 91.

† In certain problems the conic is degenerate or imaginary. Here one transformation is often sufficient and the locus can be determined by means of § 30 or § 46.

3. $4x^2 - 24xy + 11y^2 + 72x - 116y + 204 = 0.$

4. $x^2 + 24xy - 6y^2 + 4x + 48y + 34 = 0.$

5. $18x^2 - 48xy + 32y^2 - 120x + 35y + 200 = 0.$

6. $16x^2 + 24xy + 9y^2 - 60x + 80y + 200 = 0.$

7. $6x^2 + 13xy + 6y^2 - 7x - 8y + 2 = 0.$

8. $9x^2 - 24xy + 16y^2 - 3x + 4y - 6 = 0.$

9. $2x^2 + 4xy + 4y^2 + 2x + 3 = 0.$

10. $2x^2 + 6xy + 10y^2 - 2x - 6y + 19 = 0.$

11. $41x^2 + 24xy + 34y^2 - 90x + 120y + 225 = 0.$

12. $13x^2 + 10xy + 13y^2 + 16x - 16y + 16 = 0.$

13. $16x^2 + 24xy + 9y^2 - 60x - 45y + 50 = 0.$

14. $7x^2 - 48xy - 7y^2 = 0.$

15. $13x^2 + 10xy + 13y^2 - 42x + 6y - 27 = 0$

16. $x^2 + 4xy + y^2 - 2x - 10y - 11 = 0.$

17. $3x^2 + 8xy - 3y^2 - 10x - 30y + 20 = 0.$

18. $6x^2 - 4xy + 9y^2 - 40x + 30y + 55 = 0.$

19. $3x^2 - 2\sqrt{3}\,xy + y^2 - (8 + 12\sqrt{3})\,x + (12 - 8\sqrt{3})\,y - 12 = 0.$

20. The difference of the base angles of the triangle whose base joins (0, 0) and (4, 0) is 45°. Find the locus of the vertex and simplify the equation.

21. The ends of the base of a triangle are (0, 0) and (4, 0). Find the locus of the vertex if the sum of the slopes of the sides is (*a*) $\frac{7}{12}$; (*b*) $-\dfrac{2}{\sqrt{3}}$. Simplify the equation found and draw the locus.

22. Solve the previous problem when the given points are (0, 0) and (*a*, 0) and the sum of the slopes is any constant *k*. Then show that the locus is a hyperbola passing through the given points and having its axes of symmetry inclined at an angle $\frac{1}{2}\theta$, where $\tan\theta = -\dfrac{2}{k}$.

98. The Conic through Five Points. — The general equation of the second degree involves six arbitrary constants. As in the case of the circle, however, one of these can be divided out; hence, if we can express five of the coefficients in terms of the sixth, the conic is completely determined. This means that the conic is completely determined by five points, or by five other geometric conditions.

To find the equation of a conic so defined we proceed to form equations between the coefficients of the general equation and solve them. For example, suppose the conic is to pass through the points $(4, 2)$, $(2, 4)$, $(-3, 1)$, $(1, -3)$, and $(0, 0)$. Substituting these in the general equation,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

we have:
$$\begin{aligned} 16\,A + 8\,B + 4\,C + 4\,D + 2\,E + F &= 0,\\ 4\,A + 8\,B + 16\,C + 2\,D + 4\,E + F &= 0,\\ 9\,A - 3\,B + C - 3\,D + E + F &= 0,\\ A - 3\,B + 9\,C + D - 3\,E + F &= 0,\\ F &= 0. \end{aligned}$$

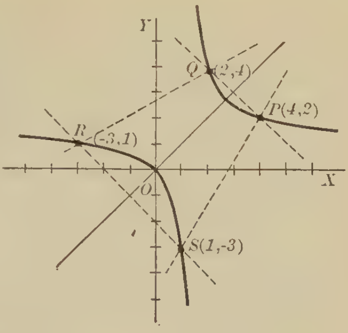
Solving these, we find $A = C = \dfrac{B}{50}$, $D = E = -\dfrac{7\,B}{5}$.

This makes the general equation, on dividing out the B and clearing of fractions:

$$x^2 + 50\,xy + y^2 - 70\,x - 70\,y = 0.$$

The value of Δ shows that the conic is a hyperbola. Its center is found by the method of § 88 to be $(\frac{35}{26}, \frac{35}{26})$. The inclination of its axes is 45°, since $\tan 2\,\theta = \infty$. This information, together with the five points given, is sufficient for drawing the curve. The graph follows.

A shorter method of solving the above problem is given by the theorem of § 45. Call the first four



points P, Q, R, and S. The equations of the lines PQ and RS are $x + y - 6 = 0$ and $x + y + 2 = 0$. Multiplying these together, the equation of the pair of lines is

$$x^2 + 2xy + y^2 - 4x - 4y - 12 = 0,$$

by § 46. Similarly the equation of the pair of lines RQ and SP is

$$15x^2 - 34xy + 15y^2 + 28x + 28y - 196 = 0.$$

Now the intersections of these two equations are the points P, Q, R, and S. Hence by § 45

$$15x^2 - 34xy + 15y^2 + 28x + 28y - 196 + k(x^2 + 2xy + y^2 - 4x - 4y - 12) = 0 \quad (a)$$

is the equation of a system of curves passing through these four points. As this is of the second degree for all values of k, they are all conics. To find the conic of this system passing through the fifth point (0, 0), substitute its coördinates for the variables. This gives

$$-196 - 12k = 0,$$

whence $k = -\frac{49}{3}$. Putting this value for k in (a) and collecting terms, we have the same result as before.

If special conditions are imposed on the conic, fewer than five points will determine it. If the conic is a parabola, $B^2 - 4AC = 0$. If the conic is symmetrical with respect to the origin, $D = E = 0$. If the axes of symmetry of the conic are parallel to the coördinate axes, $B = 0$. In the first and third cases four points are sufficient, in the second three points.

PROBLEMS

1. Find the equation of the conic passing through the following points:

(*a*) $(3, 2), (1, 2), (3, 0), (2, \frac{9}{4}), (\frac{11}{8}, 1)$;
(*b*) $(1, 6), (-3, -2), (-5, 0), (3, 4), (0, 10)$.

2. The same as Problem 1 for the following points:

(*a*) $(3, 1), (3, 5), (6, 2), (6, 10), (11, 5)$;
(*b*) $(1, 1), (-1, 2), (0, -2), (-2, -1), (3, -3)$.

3. Find the equation of the parabola passing through the following points:

(*a*) $(0, 0), (4, 0), (0, 3), (-8, 8)$;
(*b*) $(0, 1), (4, 7), (4, 1), (12, 7)$.

4. Find the equation of the parabola passing through the following points:

(*a*) $(1, 1), (4, 0), (0, 4), (9, 1)$;
(*b*) $(-3, 1), (1, -3), (2, 4), (4, 2)$.

5. Find the equation of a conic symmetrical with respect to the origin and passing through:

(*a*) $(2, 4), (3, 1), (-2, 6)$;
(*b*) $(1, 2), (-1, 1), (3, 2)$;
(*c*) $(0, 0), (6, 4), (4, 6)$.

6. Find the equation of a conic with axes of symmetry parallel to the coördinate axes and passing through:

(*a*) $(-4, 5), (6, 0), (-4, -5), (4, 3)$;
(*b*) $(7, 1), (6, -2), (7, -5), (-2, -2)$;
(*c*) $(-3, -4), (-1, -3), (1, 0), (3, 5)$.

7. Find the equation of the parabola passing through the following points and whose axis is parallel to the x-axis:

(*a*) $(-1, 2), (1, 1), (2, 3)$; (*c*) $(2, 0), (3, 1), (5, 2)$;
(*b*) $(-1, 1), (0, 0), (1, 2)$; (*d*) $(-1, 0), (0, 1), (3, 2)$.

8. Find the equation of the parabola whose axis is parallel to the y-axis and which passes through each set of points in Problem 7.

CHAPTER IX

POLAR COÖRDINATES

99. Preliminary Note. — The student who is not thoroughly familiar with trigonometry will find his work in this chapter materially lightened if he reviews the following topics:

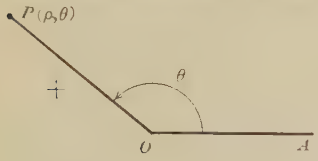
1. The signs and variation of the trigonometric functions as the angle varies from 0° to 360°.
2. Radian measure of angles.
3. The values of the functions of the special angles 0°, 90°, 180°, 270°, and multiples of 30° and 45°.
4. The formulas expressing the functions of $-\theta$, $\pi - \theta$, $\pi + \theta$, and $2\,\pi - \theta$ in terms of functions of θ.

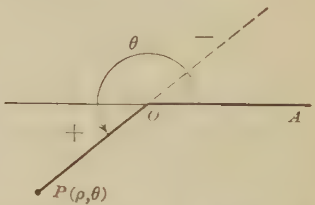
The essential information in these respects is given in the Introduction.

100. Definitions. — Some topics in analytic geometry, especially those involving motion about a point, can be better investigated by the use of *polar coördinates* than by rectangular coördinates. In the polar system the position of a point is fixed by measuring a distance and a direction instead of by the measures of two distances. This is essentially the same system as that of bearing and distance used in surveying, contrasted with that of latitude and longitude used in geography.

Choose a fixed point O as the origin, called the **pole,** and a fixed line OA through it, called the **polar axis.** Then any point P is determined if we know its distance from O

and the angle that OP makes with OA. The measures of the distance OP and the angle AOP are called the polar coördinates of P and are designated by ρ and θ.



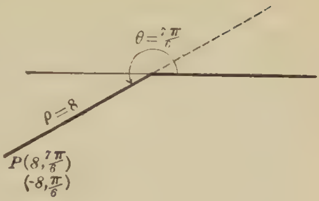


The distance ρ is called the **radius vector** of P, and θ is called the **vectorial angle.**

Polar coördinates do not obey the conventions as to direction and magnitude which we have used in the rectangular system. As in trigonometry the radius vector may be rotated indefinitely in a counter-clockwise or clockwise direction, making θ take on any positive or negative value. *Distances measured on the terminal line of θ from the pole are positive; those measured in the opposite direction, on the terminal produced, are negative.* (See the right-hand diagram above.) Hence every pair of real numbers (ρ, θ) determines one point which may be located according to the following rule.

Rule for Plotting. — *Taking the polar axis as an initial line, lay off the vectorial angle θ, counter-clockwise if positive, clockwise if negative. Then measure off the radius vector ρ, on the terminal of θ if positive, on the terminal of θ produced through the pole if negative.*

Since θ and $\theta + 2\pi$ have the same terminal line, a point may be represented by an indefinite number of pairs of coördinates. Thus in the adjoining figure we may take for the coördinates of P, $\rho = 8$, $\theta = \frac{7\pi}{6}$ or $\frac{7\pi}{6} \pm 2n\pi$. Since



the terminal line of θ produced is the terminal of $\theta \pm \pi$, a second set of coördinates is $\rho = -8$, $\theta = \frac{\pi}{6}$ or $\frac{\pi}{6} \pm 2n\pi$.

If we express θ in terms of degrees, these coördinates read as follows:

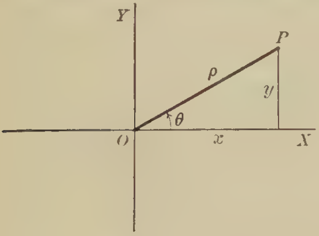
$\rho = 8$, $\theta = 210°, 570°, 930°$, etc., and $-150°, -510°$, etc.;

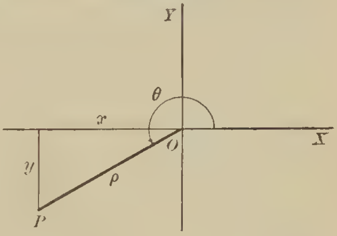
or

$\rho = -8$, $\theta = 30°, 390°, 750°$, etc., and $-330°, -690°$, etc.

Ordinarily we keep θ within the limits $\pm\pi$, or 0 and 2π.

101. Relations between Rectangular and Polar Coördinates. — Take the pole at the origin of rectangular coördinates and the polar axis as the positive half of the x-axis. From the figures and the definitions of the trigonometric functions it is evident that for $P(x, y) = P(\rho, \theta)$ in any quadrant the following formulas are true:





$$x = \rho \cos \theta, \qquad y = \rho \sin \theta, \tag{34}$$

$$\rho^2 = x^2 + y^2, \qquad \tan \theta = \frac{y}{x}. \tag{35}$$

These equations enable us to transform rectangular equations and coördinates into polar forms, and inversely.

NOTE. — In transforming the rectangular coördinates of a point into polar coördinates, care should be taken to group together the corresponding values of ρ and θ.

102. Polar Curves. — The *definitions* of equation and locus in polar are the same as those in rectangular coördinates (§ 17), if (ρ, θ) is substituted for (x, y). Since, however, a point may be represented by different pairs of polar coördinates, two different equations may have the same locus. For example, the equations $\rho = 4$ and $\rho = -4$ both have as a locus a circle of radius 4 whose center is the pole. The equation in polar coördinates is derived as in the rectangular system (§ 20). In a few cases the polar equation may be obtained best by deriving it in the rectangular form and then substituting for x and y their values in terms of ρ and θ and *vice versa.*

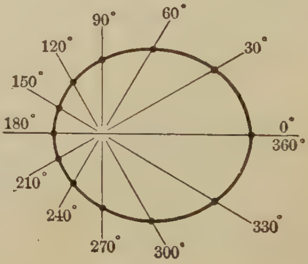
Plotting in polar coördinates resembles that in rectangular coördinates. The equation should usually be solved for ρ and a table of values formed, taking values of θ at intervals of 30° or less. If the equation involves functions of multiple angles, the values of θ should be taken so close together that the interval between successive values of the multiple is not more than 30°. For example, in plotting $\rho = 10 \cos 3\theta$, take θ at 10° intervals. When the curve has symmetry, it is usually unnecessary to carry the table through more than two quadrants. Polar plotting paper should always be used.

EXAMPLE. — Plot the ellipse $2\rho - \rho \cos \theta = 6$.

Solving, $\rho = \dfrac{6}{2 - \cos \theta}$.

TABLE OF VALUES

$\theta°$	ρ	$\theta°$	ρ
0	6.0	210	2.1
30	5.3	240	2.4
60	4.0	270	3.0
90	3.0	300	4.0
120	2.4	330	5.3
150	2.1	360	6.0
180	2.0		



PROBLEMS

1. Plot each of the following pairs of points: (*a*) $(6, \pm 30°)$; (*b*) $(\pm 10, 420°)$; (*c*) $\left(8, \frac{\pi}{6}\right)\left(8, \frac{5\pi}{6}\right)$. What symmetry has each pair of points?

2. Plot $\left(8, \frac{2\pi}{3}\right)$. Plot points symmetrical to this with respect to the pole, polar axis, and the 90° axis, and find their coördinates.

3. As in Problem 2 find the points symmetrical to:

(*a*) $\left(16, \frac{4\pi}{3}\right)$; (*b*) $\left(-6, -\frac{2\pi}{3}\right)$; (*c*) $\left(-10, -\frac{3\pi}{2}\right)$.

4. Fix $P(\rho, \theta)$ in any quadrant and find its symmetry to:

(*a*) $(\rho, -\theta)$;	(*c*) $(\rho, \pi - \theta)$;	(*e*) $(-\rho, \pi - \theta)$;
(*b*) $(-\rho, \theta)$;	(*d*) $(\rho, \pi + \theta)$;	(*f*) $(-\rho, -\theta)$.

5. Find the rectangular coördinates of the following points:

(*a*) $(16, 30°)$;	(*c*) $\left(6, \frac{\pi}{2}\right)$;	(*e*) $(-2, 45°)$;
(*b*) $\left(4, \frac{2\pi}{3}\right)$;	(*d*) $(4, 180°)$;	(*f*) $\left(-8, \frac{3\pi}{4}\right)$.

6. Find the rectangular coördinates of the following points:

(*a*) $(20, 45°)$;	(*c*) $(8, 300°)$;	(*e*) $\left(-8, -\frac{\pi}{6}\right)$;
(*b*) $(16, 210°)$;	(*d*) $\left(10, \frac{2\pi}{3}\right)$;	(*f*) $\left(12, \frac{\pi}{2}\right)$.

7. Find two pairs of polar coördinates for each of the following points and plot the point in each case:

(*a*) $(-1, \sqrt{3})$;	(*c*) $(5, -12)$;	(*e*) $(-\sqrt{2}, -\sqrt{2})$;
(*b*) $(2, -2)$;	(*d*) $(2, 2\sqrt{3})$;	(*f*) $(3, 4)$.

8. Find two pairs of polar coördinates for each of the following points and plot the point in each case:

(*a*) $(\sqrt{3}, -1)$;	(*c*) $(-4, -4\sqrt{3})$;	(*e*) $(-6, 0)$,
(*b*) $(-8, 8)$;	(*d*) $(0, -10)$;	(*f*) $(10\sqrt{3}, -10)$.

9. Plot each of the following equations by means of polar coördinates and identify each curve by transforming its equation to rectangular coördinates:

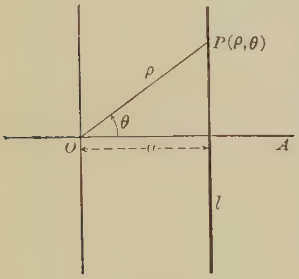
(*a*) $\rho = 4$;
(*b*) $\tan\theta = \frac{3}{4}$;
(*c*) $\rho = 10\cos\theta$;
(*d*) $\rho\cos\theta = -2$;
(*e*) $\rho = \pm 6$;
(*f*) $\tan\theta = \frac{2}{3}$;
(*g*) $\rho = 2a\sin\theta$;
(*h*) $\rho\sin\theta = 2a$;
(*i*) $\rho = \sin\theta + \cos\theta$.

10. Draw each of the following curves and transform its equation into polar coördinates:

(*a*) $y = 2x$;
(*b*) $3x + 4y = 0$;
(*c*) $x - a = 0$;
(*d*) $y = -3$;
(*e*) $x^2 + y^2 = a^2$;
(*f*) $x^2 + y^2 - 6x = 0$;
(*g*) $x^2 + y^2 + 8y = 0$;
(*h*) $x^2 - y^2 = 16$;
(*i*) $y^2 - x^2 = 16$;
(*j*) $x^2 - y^2 = a^2$;
(*k*) $2xy = a^2$;
(*l*) $2xy = -a^2$.

103. Equations of the Straight Line.

— The general equation of the straight line in polar coördinates is not as convenient as the equation in rectangular coördinates and will not be discussed. The special cases where the line is parallel or perpendicular to the polar axis or passes through the pole lead to very simple equations.



In the figure let the line l be perpendicular to the polar axis OA and have the polar intercept a. For any point $P(\rho, \theta)$ on the line it is evident that $\cos\theta = \dfrac{a}{\rho}$, whence the equation of the line is

$$\rho\cos\theta = a. \tag{36}$$

Similarly the equation of a line parallel to the polar axis, of 90° intercept a, is

$$\rho\sin\theta = a. \tag{36a}$$

For a line passing through the pole the equation is evidently

$$\theta = c. \tag{37}$$

Exercise 1. By transforming the normal form of the straight-line equation into polar coördinates show that the equation of any line in polar coördinates is

$$\rho \cos (\theta - \omega) = p.$$

104. Equations of the Circle. — As in the case of the straight line the general form of the circle equation is not often used. Several special forms, which are common, are:

Circle with center at pole, radius r: $\rho = r;$ (38)

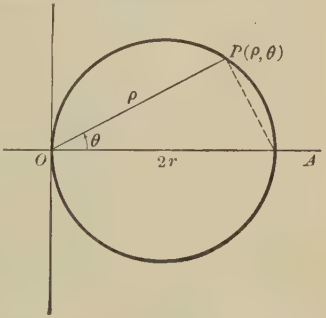
Circle with center at $(r, 0)$, *radius r:* $\rho = 2\, r \cos \theta;$ (39)

Circle with center at $\left(r, \frac{\pi}{2}\right)$, *radius r:* $\rho = 2\, r \sin \theta;$ (39*a*)

Circle through pole, with polar intercept a and 90° intercept b:

$$\rho \doteq a \cos \theta + b \sin \theta. \tag{40}$$

The first equation is obvious. The second may be readily derived from the adjoining figure, and the third from a similar figure.



The simplest way of getting the fourth equation is to observe that its rectangular equation is $x^2 + y^2 - ax - by = 0$. Transforming this into polar coördinates, we have the desired result.

Exercise 2. Derive equation (40) directly from a figure.

Hint. — Show that the radius vector is in length the sum of the projections upon it of the x- and y-intercepts of the circle.

105. Discussion of Polar Curves. — As in the case of rectangular equations, the study of polar curves is facilitated by discussion of the equation in conjunction with plotting. The topics discussed are of much the same character.

(*a*) *Intercepts.* — The intercepts on the polar and 90° axes are the values of ρ for $\theta = 0°$ and $180°$, and $90°$ and $270°$, respectively.

(*b*) *Symmetry.* — It is easy to establish that in accordance with the definitions of § 25, (ρ, θ) is symmetrical to $(-\rho, \theta)$ and $(\rho, \pi + \theta)$ with respect to the pole; to $(\rho, -\theta)$ and $(-\rho, \pi - \theta)$ with respect to the polar axis; and to $(\rho, \pi - \theta)$ and $(-\rho, -\theta)$ with respect to the 90° axis. Hence we have the following tests for symmetry:

If the substitution of

$-\rho$ *for* ρ *or* $\pi + \theta$ *for* θ *does not change the form of the equation, there is symmetry with respect to the pole;*

$-\theta$ *for* θ *or* $-\rho$ *for* ρ *and* $\pi - \theta$ *for* θ *does not change the form of the equation, there is symmetry with respect to the polar axis;*

$\pi - \theta$ *for* θ *or* $-\rho$ *for* ρ *and* $-\theta$ *for* θ *does not change the form of the equation, there is symmetry with respect to the* 90° *axis.*

Exercise 3. Prove that (ρ, θ) is symmetrical to:

(*a*) $(-\rho, \theta)$ with respect to the pole;
(*b*) $(\rho, -\theta)$ with respect to the polar axis;
(*c*) $(\rho, \pi - \theta)$ with respect to the 90° axis;
(*d*) $(\rho, \pi + \theta)$ with respect to the pole;
(*e*) $(-\rho, \pi - \theta)$ with respect to the polar axis;
(*f*) $(-\rho, -\theta)$ with respect to the 90° axis.

(*c*) *Limiting Values of* θ. — If the solution for ρ gives $\rho = a \pm \sqrt{f(\theta)}$, the equation $f(\theta) = 0$ usually determines limiting values of θ, since values of θ which make $f(\theta) < 0$ make ρ imaginary and must be excluded.

(*d*) *Extent of the Curve.* — If any values of θ make ρ infinite, these values determine the direction in which the curve extends to infinity. If ρ is never infinite, the values of θ which make ρ take on its greatest numerical value should be found. Likewise the values of θ for which $\rho = 0$, or for which ρ takes on its least numerical value, should be found.

EXAMPLE 1. — Discuss and plot the locus of the equation $\rho = \dfrac{2}{1 + \cos\theta}$.

Solution. —

(*a*) Intercepts:

θ	$0°$	$90°$	$180°$	$270°$
ρ	1	2	∞	2

(*b*) Symmetry: The locus is symmetrical with respect to the polar axis, since $\cos(-\theta) = \cos\theta$. Hence the table of values is unnecessary for $\theta > 180°$.

(*c*) Limiting values of θ: None, since no radical is involved.

(*d*) Extent of the curve: ρ is infinite when $1 + \cos\theta = 0$, *i.e.*, when $\cos\theta = -1$, or $\theta = \pi$.

ρ is evidently never 0. Its least value occurs when $1 + \cos\theta$ is greatest, *i.e.*, when $\cos\theta = 1$, or $\theta = 0$. Here $\rho = 1$.

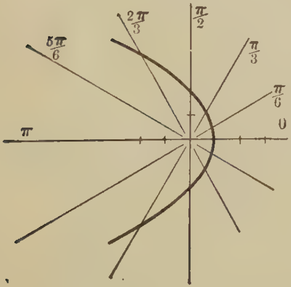


TABLE OF VALUES

θ (radians)	θ (degrees)	$1 + \cos\theta$	ρ
± 0	0	2.00	1.0
$\pm\frac{\pi}{6}$	$\pm$ 30	1.87	1.1
$\pm\frac{\pi}{3}$	$\pm$ 60	1.50	1.3
$\pm\frac{\pi}{2}$	$\pm$ 90	1.00	2.0
$\pm\frac{2\pi}{3}$	$\pm$ 120	0.50	4.0
$\pm\frac{5\pi}{6}$	$\pm$ 150	0.13	15.0
$\pm\pi$	$\pm$ 180	0.00	∞

Transformation to rectangular coördinates shows that this curve is a parabola.

Example 2. — $\rho^2 = a^2 \cos 2\theta$, or $\rho = \pm a\sqrt{\cos 2\theta}$. (Lemniscate.)

Solution. — (*a*) The intercepts on the polar axis are $\pm a$; on the 90° axis they are imaginary.

(*b*) The three tests for symmetry are satisfied, for $\cos 2(-\theta) = \cos 2\theta$, etc. Hence values of θ up to 90° only are needed.

(*c*) ρ is imaginary when $\cos 2\theta < 0$, *i.e.*, when

$$90° < 2\theta < 270°,$$

or

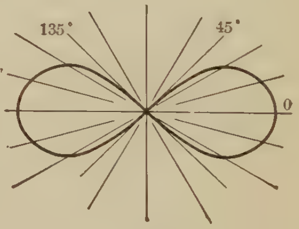
$$45° < \theta < 135°.$$

(*d*) The greatest numerical value of ρ occurs when $\cos 2\theta = 1$, or $\theta = 0$ or 180°. This value is a.

$\rho = 0$ when $\cos 2\theta = 0$, or when $2\theta = 90°, 270°$, etc., or $\theta = 45°, 135°, 225°$, and 315°.

Table of Values, $(a = 10)$

θ	2θ	$\cos 2\theta$	$\sqrt{\cos 2\theta}$	ρ
0	0	1.00	$\pm$ 1.00	$\pm$ 10
15°	30	.87	$\pm$ .93	$\pm$ 9.3
30°	60	.50	$\pm$ .71	$\pm$ 7.1
45°	90	.00	.00	0.0



Example 3. — $\rho = 1 + 2\sin\theta$. (Limaçon.)

Solution. — (*a*)

θ	0°	90°	180°	270°
ρ	1	3	1	−1

(*b*) Since $\sin(\pi - \theta) = \sin\theta$, the curve is symmetrical with respect to the 90° axis.

(*c*) No values of θ are excluded.

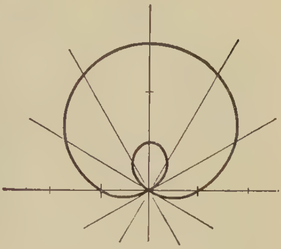
(*d*) ρ is never infinite; its greatest value occurs when $\sin\theta = 1$, or $\theta = 90°$.

$\rho = 0$ when $1 + 2\sin\theta = 0$, or $\sin\theta = -\frac{1}{2}$. Hence the curve passes through the pole when $\theta = 210°$ or 330°

TABLE OF VALUES

θ	$\sin\theta$	ρ
0	.00	1.00
30	.50	2.00
60	.87	2.73
90	1.00	3.00
120	.87	2.73
150	.50	2.00
180	.00	1.00

θ	$\sin\theta$	ρ
210	$-.50$	.00
240	$-.87$	$-.73$
270	-1.00	-1.00
300	$-.87$	$-.73$
330	$-.50$	.00
360	.00	1.00



PROBLEMS

1. Using §§ 103, 104, identify and draw the locus of each of the following:

(*a*) $\rho\cos\theta - 4 = 0$;
(*b*) $\rho\cos\theta + 4 = 0$;
(*c*) $\rho = 4$;
(*d*) $\tan\theta = \sqrt{3}$;
(*e*) $\rho - 8\sin\theta = 0$;
(*f*) $\rho + 7\cos\theta = 0$;
(*g*) $\rho^2 + 3\rho - 4 = 0$;
(*h*) $\sec^2\theta = 2$;
(*i*) $\theta = \frac{\pi}{6}$;
(*j*) $\rho = -6\sin\theta$;
(*k*) $\rho = 3\cos\theta$;
(*l*) $\rho = 6\cos\theta - 14\sin\theta$;
(*m*) $\rho + 6\sin\theta + 8\cos\theta = 0$.

2. Transform into polar coördinates:

(*a*) $Ax + By + C = 0$;
(*b*) $x^2 + y^2 + Dx + Ey + F = 0$.

Discuss* and plot the locus of each of the following:

3. (*a*) $\rho = a(1 - \cos\theta)$; (*c*) $\rho = a(1 - \sin\theta)$;
(*b*) $\rho = a(1 + \cos\theta)$; (*d*) $\rho = a(1 + \sin\theta)$. (Cardioids.)

4. (*a*) $\rho = a + b\cos\theta$, $(a < b)$;
(*b*) $\rho = a + b\cos\theta$, $(a > b)$;
(*c*) $\rho = a - b\cos\theta$, $(a < b)$;

---

* The arbitrary constants should be kept throughout the discussion, the intercepts, etc., being found in terms of the arbitrary constants. Substitution of numerical values should only be made in finding the table of values for plotting. Compare Example 2.

(*d*) $\rho = a - b\cos\theta$, $(a > b)$;
(*e*) $\rho = a + b\sin\theta$, $(a < b)$;
(*f*) $\rho = a + b\sin\theta$, $(a > b)$;
(*g*) $\rho = a - b\sin\theta$, $(a < b)$;
(*h*) $\rho = a - b\sin\theta$, $(a > b)$.

(Limaçons; the special form when $a = b$ is called a cardioid.)

5. (*a*) $\rho = a\tan^2\theta\sec\theta$; (*b*) $\rho = a\cot^2\theta\csc\theta$.
(Semi-cubical parabolas.)

6. (*a*) $\rho = a\sec\theta \pm b$, $(a < b)$;
(*b*) $\rho = a\sec\theta \pm b$, $(a = b)$;
(*c*) $\rho = a\sec\theta \pm b$, $(a > b)$;
(*d*) $\rho = a\csc\theta \pm b$, $(a < b)$;
(*e*) $\rho = a\csc\theta \pm b$, $(a = b)$;
(*f*) $\rho = a\csc\theta \pm b$, $(a > b)$.
(Conchoids of Nicomedes.)

7. (*a*) $\rho = 2a\tan\theta\sin\theta$; (*b*) $\rho = 2a\cot\theta\cos\theta$. (Cissoids.)

8. (*a*) $\rho = a\sin 2\theta$; (*b*) $\rho = a\cos 2\theta$. (Four-leafed roses.)

9. (*a*) $\rho = a\sin 3\theta$; (*b*) $\rho = a\cos 3\theta$. (Three-leafed roses.)

10. (*a*) $\rho = a\sin 5\theta$; (*b*) $\rho = a\cos 5\theta$. (Five-leafed roses.)

11. $\rho^2 = a^2\sin 2\theta$. (Lemniscate.)

12.* $\rho = a\theta$. (Spiral of Archimedes.)

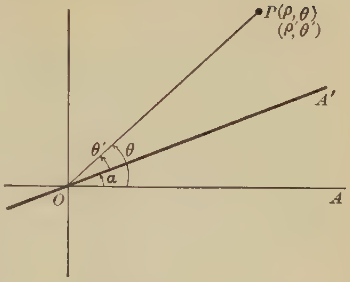
13.* $\rho\theta = a$. (Hyperbolic spiral.)

14.* $\rho^2\theta = a$. (Lituus.)

106. Rotation of Axes. — Rotation of the polar axis does not affect the value of ρ. The figure shows that if the coördinates of P with reference to OA are (ρ, θ),

---

[*] Since no trigonometric functions are involved in these equations, a discussion gives little or no help. The angle θ should be expressed in radians and enough of the curve plotted to show several revolutions of the radius vector. A proper choice of the arbitrary constant will save much labor in computing the table of values.



and to OA' are (ρ, θ'), while angle $A'OA$ is α, we have as a formula of rotation

$$\theta = \theta' + \alpha. \qquad (41)$$

By rotating the polar axis the standard equations of curves may be written in a variety of forms.

Translation of the pole is seldom necessary and is best performed by transforming the equation into rectangular coördinates.

107. Symmetrical Transformations. — Equations may also be transformed by the substitutions for symmetry. As in the case of rectangular coördinates it can be shown that the effect of such a substitution is to transform the locus into a curve symmetrical to the given locus with respect to the center or axis involved. Thus we have the following:

If in an equation substitution is made of

$$\begin{array}{r} \pi + \theta \text{ for } \theta \\ -\theta \text{ for } \theta \\ \pi - \theta \text{ for } \theta \end{array} \left\{ \begin{array}{c} \text{the locus of the new equation} \\ \text{is symmetrical to that of the} \\ \text{old with respect to the} \end{array} \right\} \begin{array}{l} \text{pole;} \\ \text{polar axis;} \\ 90^\circ \text{ axis.} \end{array}$$

PROBLEMS

1. In the following find the new equation on rotating the axes through the indicated angle, draw the curve and both axes:

(*a*) $\rho = a \tan^2 \theta \sec \theta, \alpha = 90^\circ$; (*c*) $\rho = a \sin 3\theta, \alpha = 30^\circ$;
(*b*) $\rho = a \sec \theta \pm b, \alpha = 90^\circ$; (*d*) $\rho = a \cos \theta, \alpha = 90^\circ$;
(*e*) $\rho = a + b \cos \theta, \alpha = 90^\circ$.

2. Show that the first of each of the following pairs of equations may be transformed into the second by a proper rotation of the polar axis:

(*a*) $\rho = a(1 + \cos\theta)$, $\rho = a(1 - \sin\theta)$;
(*b*) $\rho = a\sin 2\theta$, $\rho = a\cos 2\theta$;
(*c*) $\rho = a\sin 3\theta$, $\rho = a\cos 3\theta$;
(*d*) $\rho^2 = a^2\sin 2\theta$, $\rho^2 = a^2\cos 2\theta$.

3. Find the new equation when the substitution for symmetry with respect to the 90° axis is made, and draw both curves on the same axes:

(*a*) $\rho = a(1 - \cos\theta)$;
(*b*) $\rho = a + b\cos\theta$;
(*c*) $\rho = a\tan^2\theta\sec\theta$;
(*d*) $\rho = 2a\tan\theta\sec\theta$;
(*e*) $\rho = a\cos 3\theta$;
(*f*) $\rho = a\cos\theta$.

4. To each equation apply the substitution for symmetry with respect to the polar axis and draw both curves on the same axes.

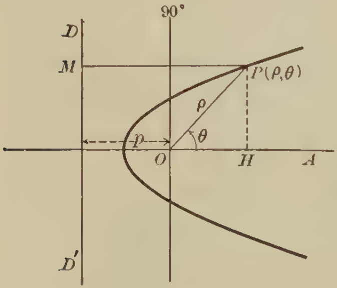
(*a*) $\rho = a(1 - \sin\theta)$;
(*b*) $\rho = a - b\sin\theta$;
(*c*) $\rho = a\sin 3\theta$;
(*d*) $\rho = a\sin\theta$.

108. Polar Equation of the Conics. — If the pole is taken at the focus and the directrix is perpendicular to the polar axis as in the figure, we have at once from the definition of a conic (§ 90), $OP = eMP$, whence

$$\rho = e(p + \rho\cos\theta),$$

which gives

$$\rho = \frac{ep}{1 - e\cos\theta}. \qquad (42)$$



If the directrix is taken parallel to the polar axis, the equation becomes

$$\rho = \frac{ep}{1 - e\sin\theta}. \qquad (42a)$$

The transformations of rotation and symmetry give eight forms of the conic equation in which the focus is at the pole and the directrix perpendicular or parallel to the polar axis. They are:

$$\rho = \frac{\pm ep}{1 \pm e \cos \theta}; \qquad \rho = \frac{\pm ep}{1 \pm e \sin \theta}.$$

In the first set the principal axis is the polar axis; in the second it is the 90° axis. To sketch a conic it is sufficient to put the equation in standard form and find the intercepts. The standard form determines the principal axis and eccentricity, and the intercepts give the vertices and the ends of the latus rectum.

By using the polar equation many of the properties of the conics may be derived more easily than by the use of the rectangular equations. Some of these are indicated in the following problems.

EXAMPLE. — Identify and draw the conic whose equation is $\rho = \dfrac{15}{2 + 4 \sin \theta}$.

Solution. — To reduce the equation to the standard form divide numerator and denominator by 2. This gives us

$$\rho = \frac{\frac{15}{2}}{1 + 2 \sin \theta}$$

and shows that the curve is a hyperbola whose eccentricity is 2 and whose principal axis is the 90° axis.

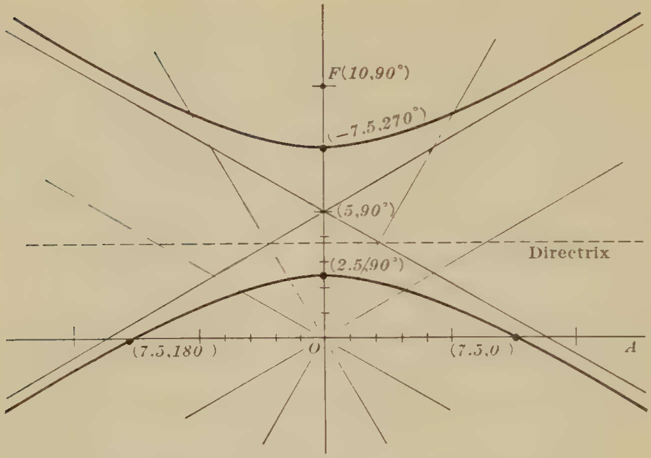
The intercepts are

0°	90°	180°	270°
7.5	2.5	7.5	−7.5

Since the pole is the focus, the latus rectum is 15.

The center is easily seen to be the point (5, 90°) and the other focus is therefore (10, 90°). Two other points are then 7.5 units to the right and left of the second focus.

Since $ep = \dfrac{15}{2}$, $p = \dfrac{15}{4}$ and the directrix is parallel to the polar axis, $\dfrac{15}{4}$ units above it.



Since ρ becomes infinite when $1 + 2\sin\theta = 0$, or $\sin\theta = -\frac{1}{2}$, the asymptotes must be parallel to the radii vectores corresponding to $\theta = 210°$ and $\theta = 330°$.

PROBLEMS

1. Transform the equation of the conic, $x^2 + y^2 = e^2(x+p)^2$, into the polar form.

2. Show that the conic is symmetrical with respect to the principal axis. Does the polar form show any other axis of symmetry?

3. Find the latus rectum of the conic.

4. Reduce to the standard polar form each of the following conics, find e and p, and draw the curve and its directrix:

(a) $\rho = \dfrac{16}{2 - \cos\theta}$;

(b) $\rho = \dfrac{24}{3 + \sin\theta}$;

(c) $\rho = \dfrac{8}{1 - 2\sin\theta}$;

(d) $\rho = \dfrac{2}{2 + 3\cos\theta}$;

(e) $\rho = \dfrac{10}{1 + \cos\theta}$;

(f) $\rho = \dfrac{9}{2 - 2\sin\theta}$;

(*g*) $\rho = \dfrac{12}{4 + \cos\theta}$;

(*h*) $\rho = \dfrac{21}{2 - 5\cos\theta}$;

(*i*) $\rho = \dfrac{20}{3 - \sin\theta}$;

(*j*) $\rho = \dfrac{16}{2 + \cos\theta}$;

(*k*) $\rho = \dfrac{12}{1 - \cos\theta}$;

(*l*) $\rho = \dfrac{20}{3 + 3\sin\theta}$;

(*m*) $\rho = \dfrac{10}{1 - \sin\theta}$;

(*n*) $\rho = \dfrac{10}{2 + 3\cos\theta}$;

(*o*) $\rho = \dfrac{10}{2 + 3\sin\theta}$;

(*p*) $\rho = \dfrac{16}{1 - 2\cos\theta}$.

5. In the ellipses of Problem 4 find the major and minor axes.

6. Show that if $\rho = \dfrac{ep}{1 - e\cos\theta}$ is the equation of a hyperbola, the inclinations of the asymptotes are given by $\cos\theta = \pm\dfrac{1}{e}$.

7. Find the polar intercepts of the conic. From these show that

(*a*) the major axis of the ellipse is $\dfrac{2\,ep}{1 - e^2}$;

(*b*) the transverse axis of the hyperbola is $\dfrac{2\,ep}{e^2 - 1}$.

8. Show that the conic never crosses the directrix.

Hint. — Using Formula 42, the equation of the directrix is $\rho\cos\theta = -p$. Points of intersection are found by solving simultaneously.

9. Derive the equation of the conic when the equation of the directrix is:

(*a*) $\rho\cos\theta = p$; (*b*) $\rho\sin\theta = p$.

10. Draw the conic with its directrix for each form:

(*a*) $\rho = \dfrac{ep}{1 + e\cos\theta}$;

(*b*) $\rho = -\dfrac{ep}{1 + e\cos\theta}$;

(*c*) $\rho = \dfrac{ep}{1 - e\cos\theta}$;

(*d*) $\rho = -\dfrac{ep}{1 - e\cos\theta}$;

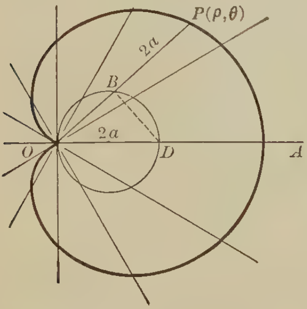
(e) $\rho = \dfrac{ep}{1 + e \sin\theta}$; (g) $\rho = \dfrac{ep}{1 - e \sin\theta}$;

(f) $\rho = -\dfrac{ep}{1 + e \sin\theta}$; (h) $\rho = -\dfrac{ep}{1 - e \sin\theta}$.

11. Show that the locus of $\rho = a \sec^2 \dfrac{\theta}{2}$ is a parabola.

12. Show that for any chord drawn through the focus of a conic the sum of the reciprocals of the lengths of the segments into which the focus divides it is $\dfrac{2}{ep}$.

109. Equations derived in Polar Coördinates. — The equation of a locus may sometimes be derived more readily in polar than in rectangular coördinates. This is especially true when the locus is described by the end of a line of varying length revolving around a fixed point. The process is substantially the same as when rectangular coördinates are used. The pole should be chosen at a convenient place, usually a fixed point about which some line in the problem revolves. Then a point (ρ, θ) satisfying the conditions is chosen, and relations between ρ, θ, and the given arbitrary constants are obtained by means (usually) of trigonometric formulas.



Example. — A chord through the end of a diameter of a circle is extended the length of the diameter. Find the locus of the end of this line.

Solution. — Choose one end of the given diameter as the pole, and the diameter as the polar axis. Let $2a$ be the diameter of the given circle.

Then from the figure:

$$OB = 2a \cos\theta, \quad \rho = OB + 2a.$$

Therefore the equation is

$$\rho = 2a(1 + \cos\theta),$$

and the required locus is a cardioid.

PROBLEMS

1. Find the locus under the conditions of the above example when the chord is extended a distance $2b$.

2. The radius of a circle whose center is the origin is prolonged a distance equal to the ordinate of its extremity. Find the locus of the end of this line.

3. Derive a formula for the distance between two points in terms of their polar coördinates.

Hint. — Use the law of cosines.

4. Derive the polar equation of the circle having the center (ρ_1, θ_1).

5. Derive the polar equation of the ellipse when the pole is at the center.

Hint. — Use the law of cosines and the fact that the sum of the focal radii is $2a$.

6. Find the locus of the vertex C of a triangle whose base OB is of length a and which has the angle $C = \frac{1}{2}$ angle O.*

Hint. — Use the law of sines and the relation $\sin 3A = 3\sin A - 4\sin^3 A$.

7. A fixed point A is at a distance a from a fixed line BC. From A a line is drawn cutting BC in D and points P and P' are chosen on this line b units from D on either side. Find the locus of P and P' as D slides along BC.

8. A tangent is drawn to a circle whose center is the origin and terminated by the x- and y-axis. Find the locus of its midpoint.

---

* Since the vertical angle C is one third of the exterior angle at B, this curve can be used to trisect any angle, and so is called the trisectrix.

9. Find the locus of the mid-points of chords drawn from the end of a fixed diameter of a circle.

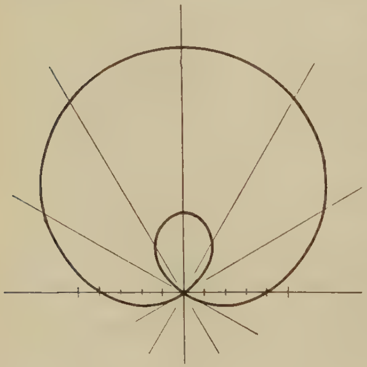
10. A line of length $2a$ has its extremities on two fixed perpendicular lines. Find the locus of the foot of a perpendicular from the intersection of the fixed lines to the line of constant length.

11. Given the circle $\rho = 2a \cos\theta$ and the line $\rho\cos\theta = 2a$. From the pole a chord OB is drawn, meeting the line in C. Find the locus of a point P on OC, if OP always equals BC.

12. The base of a triangle is in length $2a$. Find the locus of the vertex if the product of the sides is equal to a^2.

Hint. — Take the pole at the center of the base and use the law of cosines.

110. Review. — Although polar coördinates are not used in applied mathematics as frequently as are rectangular, it is desirable that the student become sufficiently familiar with the standard polar equations to be able to recognize and draw their loci without resorting to point by point plotting. The following problems are added to give practice in identifying polar curves.



EXAMPLE. Identify and sketch the locus of $\rho = 4 + 8 \sin\theta$.

Solution. — This is of the form $\rho = a + b\sin\theta$ and therefore is a limaçon symmetrical with respect to the 90° axis. The intercepts are 4, 12, 4, and -4. Plotting the corresponding points, it is apparent that this is a limaçon having a loop.

REVIEW PROBLEMS

Identify and sketch the following curves:

1. $\rho = 10 \sin \theta.$

2. $\rho = \dfrac{12}{2 - \sin \theta}.$

3. $\rho = 6 + 4 \sin \theta.$

4. $\rho = 3 \theta.$

5. $\rho = 10 \sin 2 \theta.$

6. $\rho = 6 \tan \theta \sin \theta.$

7. $\rho = 4 + 6 \sec \theta.$

8. $\rho = \dfrac{12}{1 + \cos \theta}.$

9. $\rho = 8 \cos 2 \theta.$

10. $\rho = 8 - 10 \sin \theta.$

11. $\rho \sin \theta = 10.$

12. $\rho = 4 \sec^2 \dfrac{\theta}{2}.$

13. $\rho = 10 \sin 5 \theta.$

14. $\rho \cos \theta = -5.$

15. $\rho = \dfrac{20}{3 + 7 \cos \theta}.$

16. $\rho = 5 - 10 \cos \theta.$

17. $\rho\theta = 8 \pi.$

18. $\rho^2 = 4 \cos 2 \theta.$

19. $\rho = 8 \cos 3 \theta.$

20. $\rho = 1 + \sec \theta.$

21. $\rho^2 = 16 \sin 2 \theta.$

22. $\rho = -2 \cot \theta \csc \theta.$

23. $2 \rho - \rho \cos \theta = 8.$

24. $\tan \theta = 1.$

25. $\rho = 10 \sin 3 \theta.$

26. $\rho = 8 \tan^2 \theta \sec \theta.$

27. Transform into rectangular form the standard polar equations of the following curves, as given on pages 169, 170. If several forms are given for the equation, in each case choose the first.

(*a*) the cardioid;
(*b*) the limaçon;
(*c*) the semi-cubical parabola;
(*d*) the conchoid of Nicomedes;
(*e*) the cissoid;
(*f*) the four-leafed rose;
(*g*) the lemniscate;
(*h*) the spiral of Archimedes.

CHAPTER X

HIGHER PLANE CURVES

111. Algebraic and Transcendental Equations. — The equations in Cartesian coördinates which we have hitherto treated have been *algebraic* equations, *i.e.*, have involved only integral and fractional powers of x and y.

Any equation which is not algebraic (*e.g.*, $y + \sin x = 0$) is called *transcendental*, and functions defined by such equations are called transcendental functions. The elementary equations of this class are those in which the exponential, logarithmic, trigonometric, and inverse trigonometric functions are used.

In this chapter we shall discuss a number of curves defined by transcendental equations and algebraic equations of degree higher than the second. Such curves are called *higher plane curves.*

112. The Exponential Curve. — This is defined by the exponential equation

$$y = b^x,$$

where b is any *positive* constant. The quantity b is called the *base.* If the exponent is fractional and involves even roots of b, only the positive value of the root is used.

Discussion. — For simplicity consider the case that $b > 1$.

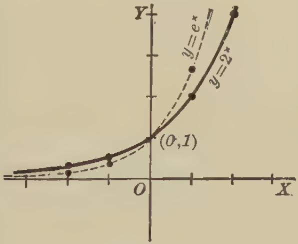
When $x = 0$, $y = b^0 = 1$. If $y = 0$, we would have $0 = b^x$, which is impossible for any value of x. Therefore the curve crosses the y-axis one unit above the origin and does not cross the x-axis.

For all values of x, positive and negative, y is positive, since any power of a positive number is positive. When $x > 0$, b^x increases with x, since $b > 1$. Therefore in the first quadrant the curve recedes to infinity away from both the x- and y-axes. Since $b^{-x} = \frac{1}{b^x}$, it is evident that the value of y for any negative value of x is the reciprocal of that for the corresponding positive value of x. We have just seen that y increases indefinitely as x approaches $+\infty$; hence y must decrease and approach 0 as x approaches $-\infty$. Thus in the second quadrant the curve approaches the x-axis as an asymptote. In the following figure the curve is plotted for two values of b.

The Base e. — The most important case of the exponential function is for the base e ($= 2.71828^+$), the base of the *natural* or *Naperian* system of logarithms. Values of various powers of e are given in any good set of tables.

In the figure the dotted line is the graph of $y = e^x$ and the heavy line that of $y = 2^x$.

x	$y = e^x$	$y = 2^x$
-2	0.135	0.250
-1	0.368	0.500
0	1.	1.
1	2.718	2.
2	7.389	4.
3	20.086	8.



113. The Logarithmic Curve. — This is the graph of the equation

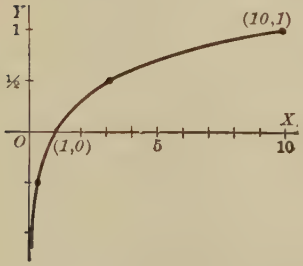
$$y = \log_b x.$$

The base b is always positive and different from 1, and usually greater than 1.

Since in algebra the logarithm of a number to a given base is defined as the exponent of that power of the base which is equal to the given number, the equation $y = \log_b x$ may be written in the form $x = b^y$. This is the exponential equation with the variables interchanged. Thus the logarithmic equation is obtained from the exponential equation $y = b^x$ by solving for x and interchanging variables. Logarithmic and exponential functions are said to be the *inverse* of each other.

The discussion of the logarithmic equation $y = \log_b x$ will therefore follow from that of the exponential equation $y = b^x$ on interchanging variables. Thus the logarithmic curve crosses the x-axis at $(1, 0)$ and does not cross the y-axis. For $x > 1$, $y > 0$ and as $x \doteq$* ∞, $y \doteq \infty$; for $x < 1$, $y < 0$ and as $x \doteq 0$, $y \doteq -\infty$. Since $x = b^y$, the abscissa x is positive for all real values of y, whether positive or negative. This is the reason for the statement that negative numbers do not have real logarithms.

The following graph is the logarithmic curve for $b = 10$.



$y = \log_{10} x$

x	y	x	y
1.0	0	.1	-1
3.1	.5	.01	-2
10	1	.001	-3
100	2	.0001	-4
etc.	etc.	etc.	etc.

The table of values may be computed by using a table of common logarithms or by writing the equation in the form $x = 10^y$ and computing values of x corresponding to various values of y. Note that the logarithmic curve is symmetrical to the exponential curve with respect to the line $y = x$.

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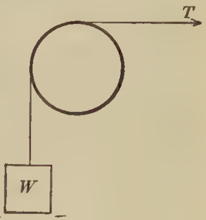
* This symbol is used for the words "approaches as a limit."

114. Applications. — The exponential and logarithmic curves show clearly the method of variation of these functions. Thus for $b > 1$, the exponential function increases with great rapidity as x approaches ∞, while the logarithmic function increases very slowly.

Exponential and logarithmic functions of various kinds constantly appear in higher mathematics, especially in its application to other sciences. For example, the adjoining figure represents a weight W suspended by a rope wrapped several times about a wooden beam and kept from falling by a tension T. The relation between W and T is given by the equation

$$W = Te^{\mu x},$$

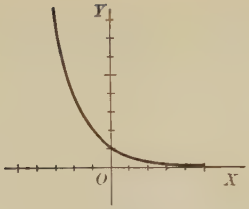
where x represents the number of times the rope is wound around the beam and μ is a constant depending on the friction between the rope and the beam. For a hemp rope on smooth oak $\mu = 3.34$, nearly. (See also § 165.)



A knowledge of the properties of the simple exponential and logarithmic functions of §§ 112, 113 may be employed advantageously in plotting more complicated functions, as the following examples show.

Example 1. — Plot $y = \left(\frac{1}{2}\right)^x$.

Solution. — By the properties of exponents this equation may be written in the form $y = 2^{-x}$. The graph is therefore symmetrical to that of $y = 2^x$ with respect to the y-axis. The table of values confirms this.

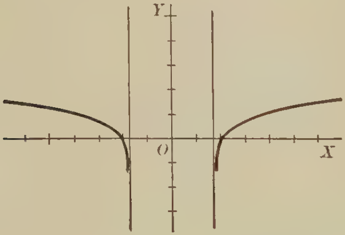


x	y
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

Example 2. — Plot $y = \log_{10}(x^2 - 3)$.

Solution. — When $x = 0$, y is not defined. When $y = 0$, $x^2 - 3 = 1$, since $0 = \log 1$. Hence the x-intercepts are ± 2. Obviously there is symmetry with respect to the y-axis.

Considering the branch where x is positive, we see that y is positive when $x^2 - 3 > 1$ or $x > 2$; and that y is negative when $x < 2$. Since the logarithm of a number approaches $-\infty$ when the number approaches 0, y approaches $-\infty$ when $x^2 - 3$ approaches 0, or when x approaches $\sqrt{3}$. Thus there is a vertical asymptote at $x = \sqrt{3}$. The other branch can be drawn by symmetry.



x	y
± 1.8	$-.62$
$\pm 2.$	0.00
$\pm 3.$	0.78
$\pm 4.$	1.11
$\pm 6.$	1.52

PROBLEMS

1. Plot on the same coördinate axes the graphs of $y = b^x$ for $b = 1, 2, 3$, and 10.

2. Plot on the same coördinate axes the graphs of $y = b^x$ for $b = 1, \frac{1}{2}$, and $\frac{1}{4}$.

3. Plot on the same coördinate axes the logarithmic curve for the bases 10 and e.

4. Plot the graphs of:

(a) $y = 3^{-x}$; (f) $y = e^{-.2x}$;
(b) $y = 4^{-x}$; (g) $y = 0.1\,e^{.2x}$;
(c) $y = (\frac{1}{3})^x$; (h) $y = xe^x$;
(d) $y = e^{-x}$; (i) $y = e^{-x^2}$;
(e) $y = e^{-.3x}$; (j) $y = x^x$.

5. By using logarithms solve each of the equations in Problem 4 for x in terms of y.

6. Plot the graphs of:

(a) $y = \log_2 x$; (e) $y = \log_{10}(4 - x)$;
(b) $y = \log_{10} x^3$; (f) $y = 3\log_e x$;
(c) $y = \log_{10}(x - 4)$; (g) $y = \log_e(1 + x^2)$;
(d) $y = \log_{10}(x + 4)$; (h) $y = \log_{10}(10 - x^2)$.

7. By using the definition of a logarithm solve each of the equations in Problem 6 for x in terms of y.

8. Discuss the following equations for symmetry and plot their graphs:

(a) $y = \dfrac{e^x + e^{-x}}{2}$. This is called the *hyperbolic cosine* and is written $y = \cosh x$;

(b) $y = \dfrac{e^x - e^{-x}}{2}$, or $y = \sinh x$;

(c) $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$. (The *catenary*.*)

9. Solve each of the equations in Problem 8 for x in terms of y.

10. What change takes place in the exponential curve as b decreases and approaches 1? as b increases and approaches ∞?

11. Plot the curve $y = 3^x$ and estimate the logarithms to the base 3 of the integers from 2 to 10.

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* This is the form assumed by a perfectly flexible cord or chain suspended between two points.

12. Plot the graph of the function $W = Te^{\mu x}$, where $T = 100$ lbs. and $\mu = 1.1$. Estimate from the graph how many turns of the rope would be required for this tension to support 1 ton. 2 tons. 10 tons.

Hint. — This value of μ makes $e^{\mu} = 3$, nearly

115. Periodic Functions. — *A periodic function of a variable is a function whose values are repeated at definite intervals as the variable increases.* Cosine x is an example. It passes successively through all values from $+1$ to -1 and back to $+1$ as the angle increases from 0 to 2π, and repeats these values in the same order as often as the angle x increases by 2π; or n times for a change of $2n\pi$ in the angle.

The interval of repetition is called the *period* of the function. If k is the period of $f(x)$, then $f(x+k) = f(x)$. The maximum numerical value of the function is called the *amplitude*. The table illustrates these definitions for several functions.

$f(x)$	PERIOD	AMPLITUDE
$\sin x$	2π	1
$\cos x$	2π	1
$\tan x$	π	∞
$\sin 2x$	π	1
$\cos \frac{x}{n}$	$2n\pi$	1

116. The Sine Curve. — This is plotted from the equation $y = \sin x$, where the values of x are expressed in radians.

We may assume values for x at any small interval, as $\frac{\pi}{6}$, and take the corresponding values of $\sin x$ from a table of natural sines.

In the accompanying table the values of x in degrees are found in column 2 and their circular measures to be used in plotting are in column 1. The circular measure of x is the abscissa and the value of its sine is the ordinate.

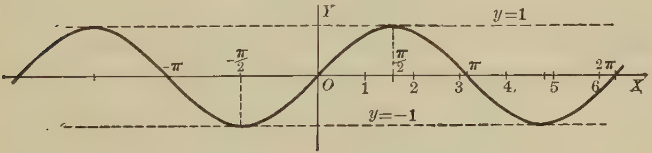
After the values from the table up to 2π have been plotted the curve is constructed through one period, and the same values of y may be used to construct it through successive periods.

x (radians)	x (deg.)	$\sin x$	$\cos x$
0	0	.00	1.00
$\frac{\pi}{6} = .52$	30	.50	.87
$\frac{\pi}{3} = 1.05$	60	.87	.50
$\frac{\pi}{2} = 1.57$	90	1.00	.00
$\frac{2\pi}{3} = 2.09$	120	.87	−.50
$\frac{5\pi}{6} = 2.62$	150	.50	−.87
$\pi = 3.14$	180	.00	−1.00
3.66	210	−.50	−.87
4.19	240	−.87	−.50
4.71	270	−1.00	.00
5.24	300	−.87	.50
5.76	330	−.50	.87
$2\pi = 6.28$	360	.00	1.00
6.81	390	.50	.87
7.33	420	.87	.50
etc.	etc.	etc.	etc.

Since the angle x may be taken negative as well as positive, the curve may be extended indefinitely in the negative direction. If we substitute $(-x, -y)$ for (x, y) in the equation $y = \sin x$, we have

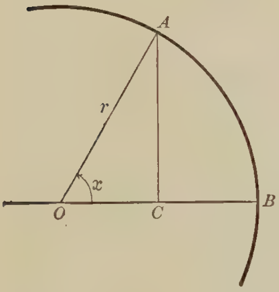
$$-y = \sin(-x) = -\sin x,$$

which may be reduced to the given equation by multiplying through by -1. Hence the curve is symmetrical with respect to the origin.



117. Circular Measure. — In plotting $y = \sin x$ we could measure x in degrees and use any convenient unit along the x-axis for a degree. If this is done, however, certain advantages attached to measuring angles in radians are lost.

Among these is the fact that when x is measured in radians, both x and $y = \sin x$ can be expressed as linear magnitudes on the same scale. For, if a central angle is measured in radians, we learn from trigonometry that



$$\text{angle} = \frac{\text{arc}}{\text{radius}},$$

or

$$x = \frac{AB}{r}.$$

$$\text{But } y = \sin x = \frac{AC}{r}.$$

Hence as the angle x is generated by rotating the side OA from the initial position OB, the lengths of AB and AC *in terms of the radius as a unit* are the values of x and $\sin x$ given in columns 1 and 3 of the table of values in § 116. This is only true where x is expressed in circular measure.

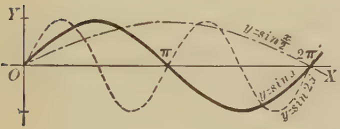
118. The Cosine Curve. — The curve of cosines is plotted from the equation $y = \cos x$. This function has the same period and amplitude as the sine function, and in fact the cosine curve has the same shape as the sine curve; but it passes through (0, 1) instead of the origin.

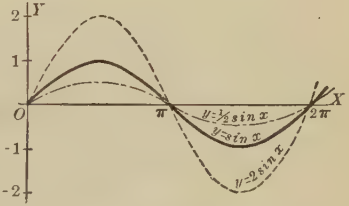
This is true because the cosine of any angle is equal to the sine of an angle 90° greater, or $\cos x = \sin (90° + x)$. Hence the table of sines can be used for plotting the cosine curve by moving the whole column of sine values up 90°, so that the sine curve becomes the cosine curve when the origin is advanced to the point $\left(\frac{\pi}{2}, 0\right)$.

119. Multiple Angles. — Consider $y = \sin\ nx$. *The multiple n divides the period of the function by n, but does not alter the amplitude.* To prove this, let us recall from § 115 that k is a period of $f(x)$ if $f(x + k) = f(x)$. We must show then that, if $f(x) = \sin nx$, $f\left(x + \frac{2\pi}{n}\right) = f(x)$. Substituting, we have

$$\sin n\left(x + \frac{2\pi}{n}\right) = \sin (nx + 2\pi) = \sin nx.$$

Hence $\frac{2\pi}{n}$ is the period of $\sin nx$. That the amplitude is unaltered is obvious.





Compare this with $y = n \sin x$, where the period is unaltered, but the amplitude is multiplied by n. All of this appears clearly in the figures. Thus we have

for $y = \sin x$, the period is 2π and $y = 1$, when $x = 90°$;
for $y = \sin 2x$, the period is π and $y = 1$, when $x = 45°$;
for $y = 2 \sin x$, the period is 2π and $y = 2$, when $x = 90°$;
for $y = 3 \sin \pi x$, the period is 2 and $y = 3$, when $x = \frac{1}{2}$.

By investigating the period we can quickly sketch the graphs of equations of the type $y = a \sin nx$, $y = a \cos nx$, etc. For we need only to determine the period and mark the points at the quarter periods, since the general shape of the curve is the same as that of the simple trigonometric curves.

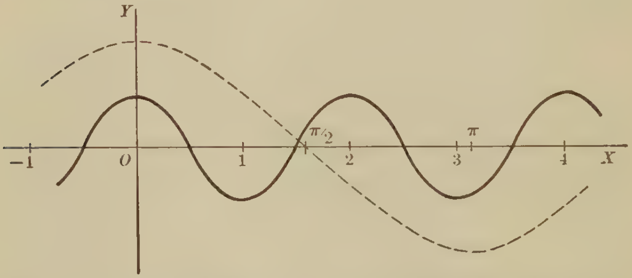
Example. — Sketch the graph of $y = \frac{1}{2} \cos \pi x$ through two periods.

Solution. — The period is $\frac{2\pi}{\pi} = 2$. The values at the quarter periods include the x-intercepts and the largest and smallest values of y. They are

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
y	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0

, etc.

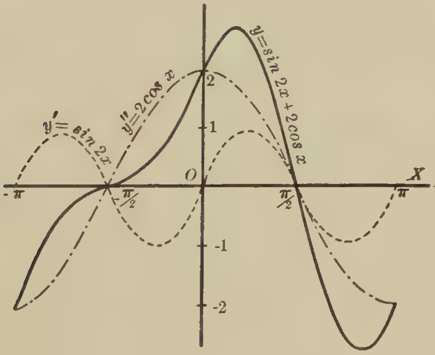
The graph of $y = \cos x$ is also shown in the figure as a dotted line for comparison.



120. Sum of Functions. — Sometimes it is necessary to plot functions which are the sum of two or more trigonometric functions. In calculating the table of values the interval between successive values of x should not be greater than $\frac{1}{12}$ of the smallest period involved, although a fair sketch is possible if $\frac{1}{4}$ of the smallest period is used. Time is saved by writing the values of the various terms in parallel columns and adding corresponding values to get the various ordinates. Thus in plotting $y = \sin 2x + 2 \cos x$, we observe that the period of $\sin 2x$ is π and take our values of x at intervals of $\frac{\pi}{12} = 15°$, obtaining the following table:

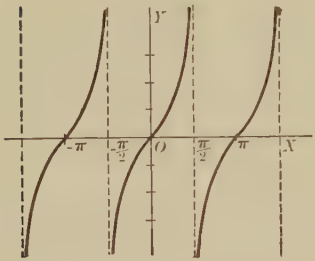
x°	x (rad)	$\sin 2x$	$2\cos x$	y
0	0	0.0000	2.0000	2.0000
15	.2618	0.5000	1.9318	2.4318
30	.5236	0.8660	1.7320	2.5980
45	.7854	1.0000	1.4142	2.4142
etc.	etc.	etc.	etc.	etc.

An alternative method is to plot the curves $y' = \sin 2x$ and $y'' = 2 \cos x$ on the same axes and obtain the sum of y' and y'' by measurement. The figure illustrates this method.



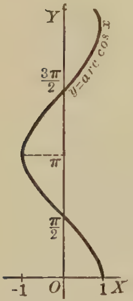
121. The Tangent Curve. — The curve of tangents is plotted from the equation $y = \tan x$. It passes through the origin and its period is π. The value of y is infinite when x is an odd multiple of $\frac{\pi}{2}$, and the curve passes through all values of y between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, and so on at intervals of π, the period.

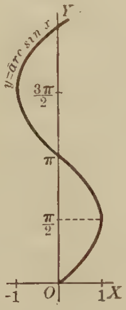
The table of values follows.



x	$\tan x$
$-\frac{\pi}{2}$	$-\infty$
$-\frac{\pi}{3}$	-1.73
$-\frac{\pi}{6}$	$-.58$
0	0
$\frac{\pi}{6}$	$.58$
$\frac{\pi}{3}$	1.73
$\frac{\pi}{2}$	∞

122. Inverse Trigonometric Functions. — The relation expressed in the inverse notation by $y = \sin^{-1} x$ reads in direct notation $x = \sin y$.





$$y = \cos^{-1} x = \text{arc} \cos x.$$
$$y = \sin^{-1} x = \text{arc} \sin x.$$

The curve of inverse sines is therefore the same as the curve of sines in shape, period, and amplitude, but the axes are interchanged.

The notation arc sin x is preferable to $\sin^{-1} x$.

PROBLEMS

1. Plot the graphs of the following, letting x range from -2π to 2π:

(a) $y = \cos x$; (c) $y = \sec x$;
(b) $y = \cot x$; (d) $y = \csc x$.

2. In the following find the period, make a table of values extending through two periods, taking x at intervals of a quarter period, and draw the graph:

(a) $y = \sin 3x$;
(b) $y = \cos 5x$;
(c) $y = \tan \frac{x}{2}$;
(d) $y = \frac{1}{3} \sec 3x$;
(e) $y = \frac{3}{2} \cos \frac{2}{3} x$;
(f) $y = 2 \cos \pi x$;
(g) $y = 2 \sin \frac{\pi x}{2}$;
(h) $y = .1 \sin 10 \pi x$.

3. Plot the graph of:

(a) $y = \text{arc} \tan x$;
(b) $y = \text{arc} \sec x$;
(c) $y = \text{arc} \cot x$;
(d) $y = \text{arc} \csc x$;
(e) $y = 2 \, \text{arc} \cos 2x$;
(f) $y = 3 \, \text{arc} \sin \frac{x}{3}$;
(g) $y = \frac{2}{\pi} \text{arc} \cos x$.

4. Draw on the same axes the graphs of:

(a) $\begin{cases} y = \sin x, \\ y = \csc x; \end{cases}$ (b) $\begin{cases} y = \cos x, \\ y = \sec x; \end{cases}$ (c) $\begin{cases} y = \tan x, \\ y = \cot x. \end{cases}$

What relation holds between corresponding ordinates of each pair of curves?

5. Draw each of the following curves. For any value of x, draw ordinates at $\pm x$, $\pi \pm x$, $2\pi \pm x$, and then write the trigonometric formulas suggested by the figure.

(a) $y = \sin x$;
(b) $y = \cos x$;
(c) $y = \tan x$;
(d) $y = \cot x$;
(e) $y = \sec x$;
(f) $y = \csc x$.

6. Plot the graphs of the following equations for the range of values indicated:*

(a) $y = \sin x + \cos x$, $(0 \text{ to } 2\pi)$;
(b) $y = \cos x - \sin x$, $(0 \text{ to } 2\pi)$;
(c) $y = 2 \sin x + \cos \frac{x}{2}$, $(0 \text{ to } 4\pi)$;
(d) $y = \cos 2x - \sin x$, $(0 \text{ to } 2\pi)$;

---

[*] In some of these problems considerable labor is saved by using a table giving the trigonometric functions of the angle expressed in radians. When such a table is not available it is often convenient to choose values of the angle at intervals of 0.2 or 0.5 radians and find the various functions involved by converting the radians into degrees.

(*e*) $y = \cos x + \sin 2x, (0 \text{ to } 2\pi);$
(*f*) $y = x + 2 \sin x, (0 \text{ to } 3\pi);$
(*g*) $y = \sec x - x, \left(-\frac{\pi}{2} \text{ to } 2\pi\right);$
(*h*) $y = \frac{x}{2} + \cos x, (0 \text{ to } 2\pi);$
(*i*) $y = x - \sin \frac{\pi x}{2}, (0 \text{ to } 6);$
(*j*) $y = x \sin x, (0 \text{ to } 3\pi);$
(*k*) $y = x \cos \frac{x}{2}, (-\pi \text{ to } 2\pi);$
(*l*) $y = x^2 \sin x, (-2\pi \text{ to } 2\pi);$
(*m*) $y = e^x \sin x, (0 \text{ to } 2\pi);$
(*n*) $y = e^x \cos x, (0 \text{ to } 2\pi);$
(*o*) $y = e^{-x} \sin x, (0 \text{ to } 2\pi);$
(*p*) $y = e^{-x} \cos x, (0 \text{ to } 2\pi).$

7. Determine which of the functions of Problem 6 are periodic and find the periods.

123. Parametric Equations. — In some curves the coordinates are functions of a third variable such that the conditions determining the curve may be conveniently expressed by two equations between these three variables instead of by a single equation in x and y. This third variable is called a *parameter*, and the two equations *parametric equations* of a curve. If t is the parameter, the parametric equations of the curve are usually of the form $x = f(t)$, $y = g(t)$.

The parameter usually represents some geometric magnitude, or the time during which the point tracing the curve has been in motion; but it may be chosen in any manner that is convenient. Thus for the semi-cubical parabola $y^2 = ax^3$, we may assume a parameter t and write $x = at^2$, $y = a^2t^3$, since by eliminating t between these equations we obtain the original form.

To plot the graph of such equations we take various values of the parameter and compute the corresponding values of x and y. Then plot the points (x, y) as usual. The values of the parameter appear only in the table of values, and not in locating the point.

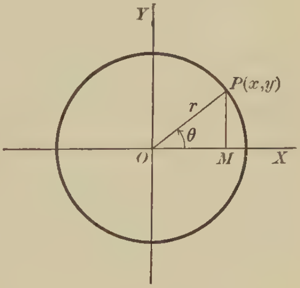
124. Parametric Equations of the Circle. — Let us take the origin at the center of the circle. Then the circle is generated by rotating OP. Taking θ as the parameter, we have at once from the figure and the definitions of the sine and cosine:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

These are the parametric equations of the circle. From the table of values below points on the curve are (10, 0), (8.67, 5), (5, 8.67), (0, 10), (−5, 8.67), (−8.67, 5), (−10, 0), (−8.67, −5), etc.

TABLE OF VALUES FOR $r = 10$

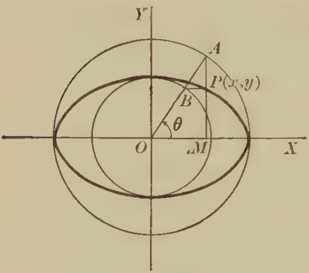
θ	x	y
0	10.00	0
30	8.67	5.00
60	5.00	8.67
90	0	10.00
120	−5.00	8.67
150	−8.67	5.00
180	−10.00	0
210	−8.67	−5.00



Note that if we eliminate θ between the parametric equations, we get the standard form $x^2 + y^2 = r^2$.

125. Parametric Equations of the Ellipse. — Suppose that we have a circle of radius a and center O, in which

OA is any radius and AM a perpendicular from A upon a fixed diameter. B is a point on the radius at a distance b from O and P is the projection of B on AM. It is required to find the locus of P.



Take the fixed diameter as the x-axis and the center as the origin. Let (x, y) be the coördinates of P and $\theta = \angle MOA$ be the parameter. Then we have at once from the figure,

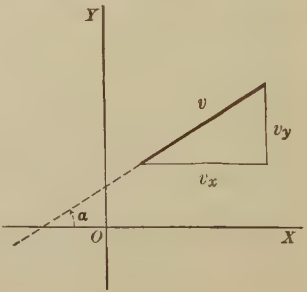
$$x = OM = OA \cos\theta = a\cos\theta,$$
$$y = MP = OB \sin\theta = b\sin\theta.$$

These are the parametric equations of the locus. It remains to show that the curve is an ellipse. Solving the equations for $\cos\theta$ and $\sin\theta$ respectively, squaring and adding we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1,$$

the equation of the ellipse.

The circles described by B and A are called the *auxiliary circles* of the ellipse. By drawing them on coördinate paper, the various positions of P can be readily found and any ellipse of given semi-axes plotted with accuracy.

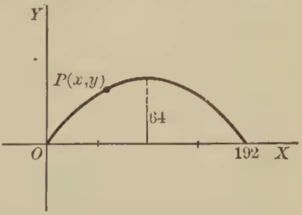


126. Path of a Projectile. — If an object moves with a velocity of v feet per second along a line inclined to the x-axis at an angle α, it is clear from the adjoining figure that at the end of each second its distance from the y-axis has been increased by $v_x = v \cos \alpha$ feet, and that from the x-axis by $v_y = v \sin \alpha$ feet. These quantities, v_x and v_y are called the x- and *y-components*, respectively, of the velocity, and when multiplied by the time give the total displacement parallel to the x- and y-axes.

In the case when the motion is in a vertical plane, the vertical velocity does not remain constant, but is affected by the force of gravity. In this case we find in mechanics that the height is given by the formula $h = v_y t - 16\,t^2$, where h is the height in *feet*, t the time in *seconds*, and v_y the vertical component of the initial velocity in *feet per second*.*

Using these principles, let us solve the problem: Find the path of a projectile discharged at an angle $\alpha = \text{arc} \cos \frac{3}{5}$ with an initial velocity of 80 feet per second.

Taking the point of projection as the origin, the y-axis vertical to the earth's surface and the x-axis horizontal in the direction of projection, let $P(x, y)$ be the position of the projectile after t seconds.



The horizontal component of the initial velocity is $v_x = 80 \cos \alpha = 80 \cdot \frac{3}{5} = 48$; the vertical component is $v_y = 80 \sin \alpha = 80 \cdot \frac{4}{5} = 64$. As the horizontal velocity remains constant,

$$x = v_x t = 48\,t. \qquad (a)$$

Using the above formula for vertical motion, we have

$$y = v_y t - 16\,t^2 = 64\,t - 16\,t^2. \qquad (b)$$

Equations (a) and (b) are the parametric equations of the locus, and the graph can now be plotted from these equations. To identify the curve, eliminate t and we have

$$y = \tfrac{4}{3}\,x - \tfrac{1}{144}\,x^2.$$

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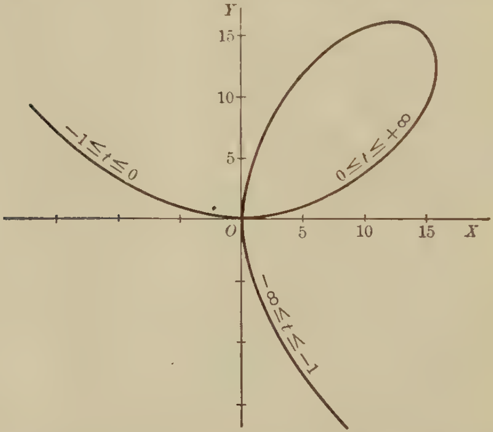
* In this and all that follows, we disregard air resistance. The results are therefore first approximations to the true ones.

Hence, the locus is a parabola with a vertical axis of symmetry.

If we set $y = 0$ in equation (b) we get $t = 0$ or 4. Hence the *time of flight* is 4 seconds. Setting $t = 4$ in equation (a) gives the *range* $x = 192$ feet. The abscissa of the vertex of the parabola is then $\frac{1}{2} \cdot 192 = 96$. For this value of x, $t = 2$ and $y = 128 - 64 = 64$. Thus the *greatest height* reached by the projectile is 64 feet.

PROBLEMS

1. The equations of the *folium of Descartes* are $x = \dfrac{3\,at}{1 + t^3}$, $y = \dfrac{3\,at^2}{1 + t^3}$. Make a table of values and plot the graph.



Solution. — The parameter t ranges in value from $-\infty$ to $+\infty$. In general, spacing the values of the parameter evenly will not give sufficient points to plot the curve unless a very large number are taken. In this particular example we find by trial that we need a large number of values between -1 and $+1$. After making the table of values for $a = 10$ we plot the points and join them in the order of the values of t. The range of values of t for the parts of the curve in the various quadrants is indicated on the graph.

Table of Values

t	x		t	x	y
$\pm\infty$	0	0	0	0	0
-10	0.30	-3.00	0.1	3.00	0.30
-5	1.21	-6.05	0.2	5.95	1.19
-2	8.57	-17.14	0.5	13.33	6.67
-1.5	18.95	-28.42	1	15	15
-1	$\pm\infty$	$\mp\infty$	1.5	10.29	15.43
-0.5	-17.14	8.57	2	6.67	13.33
-0.2	-6.05	1.21	5	1.19	5.95
-0.1	-3.00	0.30	10	0.30	3.00

2. Make tables of values and plot the graphs of the following parametric equations:

(a) $x = 3t, y = \dfrac{12}{t}$;
(b) $x = t^2, y = 2t$;
(c) $x = t + 2, y = \dfrac{t^3}{2}$;
(d) $x = 4t^2, y = 4t - t^3$;
(e) $x = t^2 + 1, y = t^3 - 1$;
(f) $x = at, y = b\sqrt{1 - t^2}$;
(g) $x = 20 \cos \theta, y = 10 \sin \theta$;
(h) $x = 2 + 5 \cos \theta, y = 5 \sin \theta - 2$;
(i) $x = a \sec \theta, y = a \tan \theta$;
(j) $x = 5 \csc \theta, y = 2 \cot \theta$;
(k) $x = a \cos^3 \theta, y = b \sin^3 \theta$;
(l) $x = a \cos \theta, y = b \sin^3 \theta$;
(m) $x = a\theta, y = a(1 - \cos \theta)$;
(n) $x = a \sec^2 \theta, y = 2a \tan \theta$.

3. Eliminate the parameter in each pair of equations in Problem 2.

4. The equation of the *folium of Descartes* is $x^3 + y^3 = 3axy$. By substituting $y = tx$ obtain the parametric equations of Problem 1.

5. A straight line has the x-intercept 10 and the inclination 30°. Find its parametric equations if the parameter t is taken as the distance of the tracing point from the intersection with the x-axis.

6. Find the parametric equations of a circle whose center is $(4, -3)$ and whose radius is 12.

7. Find the parametric equations of the path of a projectile, its range, and the highest point reached from the following data.

Use the method of § 126, but do not use the results there obtained as formulas.

(*a*) initial velocity 576 feet per second, angle of projection 45°;

(*b*) initial velocity 400 feet per second, angle of projection 30°;

(*c*) initial velocity 800 feet per second, angle of projection arc sin $\frac{3}{5}$;

(*d*) initial velocity 200 feet per second, angle of projection arc sin $\frac{4}{5}$.

8. Eliminate the parameter from the results in Problem 7.

9. In Problem 7(*a*) the projectile is fired up a hill of slope $\frac{1}{2}$. Find how far up the hill it will strike the ground.

10. Find the parametric equations of the path of a projectile if the initial velocity is v and the angle of projection is α.

11. Eliminate the parameter in the result of Problem 10.

12. In Problem 10 find the range and the greatest height. For what value of α will the range be greatest?

13. A point moves on a spoke of a wheel from the hub toward the rim at a rate of 4 feet per minute. The wheel makes 2 revolutions per minute. Find the equation in polar coördinates of the path described by the point.

14. A point moves around a circle of radius 10 feet 50 times per minute. Write the parametric equations of the circle, using the time during which the point has been in motion as a parameter.

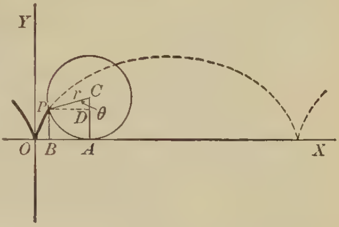
15. The parametric equations of the path of a point are

$$x = a \cos kt,$$
$$y = 0,$$

where x is measured in feet, t is measured in seconds and a and k are constants. Plot the locus. How often does the point pass through the origin?

127. The Cycloid. — If a circle rolls along a straight line the curve traced by a point on the circumference is called a *cycloid*. The straight line is called the base.

As the circle rolls it describes the length of the circumference on the base at each revolution, and a proportional distance for any part of a revolution. Hence the point of the circle touching the base is always at a distance from the starting point equal to the length of the arc subtending θ.



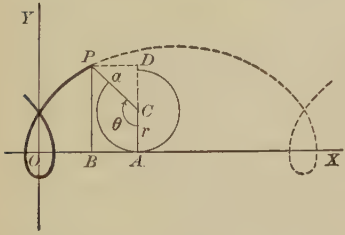
In the figure the tracing point P starts from the point of contact O, which is taken as the origin, and the base is taken as the x-axis.

Call the angle which the radius to the tracing point makes with the vertical, θ. Then OA = arc $PA = r\theta$, since arc = angle × radius. This gives

$$x = OA - BA = r\theta - r\sin\theta,$$
$$y = AC - DC = r - r\cos\theta,$$

which are the parametric equations of the cycloid.

The Prolate and Curtate Cycloids. If the tracing point is taken anywhere on the radius of the rolling circle or the radius produced, the curve is called the *prolate cycloid* when P is without the circle and the *curtate cycloid* when P is within. The term *trochoid* is also used for both prolate and curtate cycloids.



Let a be the distance from the center to the tracing point. The parametric equations of both prolate and curtate cycloids are

$$x = r\theta - a\sin\theta,$$
$$y = r - a\cos\theta.$$

They are derived in precisely the same manner as those of the ordinary cycloid.

PROBLEMS

1. Sketch the path of a point on a 30-inch automobile wheel for two revolutions. Find the length of the base of one arch of the cycloid and the coördinates of the tracing point when the wheel has made one third, one half, and two thirds of a revolution.

2. Plot the cycloid for $r = 10$, forming a table of values for θ, x, and y.

3. Eliminate θ between the parametric equations of the cycloid.

Hint. — $\cos\theta = \frac{r - y}{r}$ and $\text{vers}\,\theta = 1 - \cos\theta$.

4. Write out in full the derivation of the parametric equations of (*a*) the curtate cycloid; (*b*) the prolate cycloid.

5. Plot the curtate cycloid through two periods when $r = 8$ and $a = 5$.

6. Plot the prolate cycloid for $r = 10$ and $a = 12$.

7. In the prolate cycloid of Problem 6 find the values of the y-intercepts and of two x-intercepts.

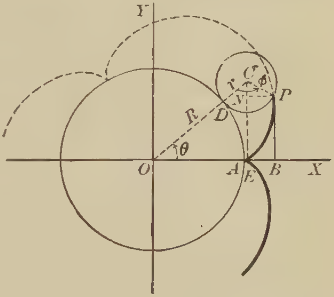
8. Show from the equations of the cycloid that y is a periodic function of x, of period $2\pi r$.

9. Show that the first arch of the cycloid is symmetrical with respect to the line $x = \pi r$.

Hint. — Show that for any value of y, as y_1, $\theta = \theta_1$, or $2\pi - \theta_1$. From this get two values of x, x_1 and x_2, and show that $x_2 - \pi r = \pi r - x_1$. Then apply the definition of symmetry.

128. The Epicycloid and Hypocycloid. — If a circle rolls upon the circumference of a fixed circle, the path described by a given point on the circumference of the generating circle is called an *epicycloid* if the generating circle rolls on the outside of the fixed circle, and a *hypocycloid* if it rolls inside.

Let the tracing point start from A and the generating circle move on the outside until its contact with the base is at D. Thus the arc AP of an epicycloid is described and the arc PD has rolled over the arc AD of the base.



Let the angle which the generating radius CP makes with the line of centers be ϕ, and the angle of this line CO with the x-axis be θ.

Using circular measure, since arc AD = arc PD,

$$R\theta = r\phi. \tag{a}$$

Now CPN is the supplement of $\theta + \phi$,

$$\text{and} \qquad \therefore \sin CPN = \sin(\theta + \phi),$$

$$\cos CPN = -\cos(\theta + \phi).$$

$$\begin{aligned}\text{But} \qquad x &= OB = OE + NP \\ &= (R + r)\cos\theta + r\cos CPN \\ &= (R + r)\cos\theta - r\cos(\theta + \phi),\end{aligned}$$

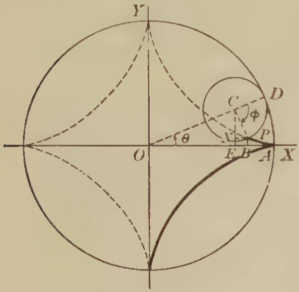
$$\begin{aligned}\text{and} \qquad y &= BP = EC - NC \\ &= (R + r)\sin\theta - r\sin(\theta + \phi).\end{aligned}$$

These equations contain two parameters, θ and ϕ. ϕ may be eliminated by substituting its value from (a), which gives

$$x = (R + r)\cos\theta - r\cos\frac{R + r}{r}\theta,$$

$$y = (R + r)\sin\theta - r\sin\frac{R + r}{r}\theta.$$

If r is replaced by $-r$ we have the equations of the hypocycloid:



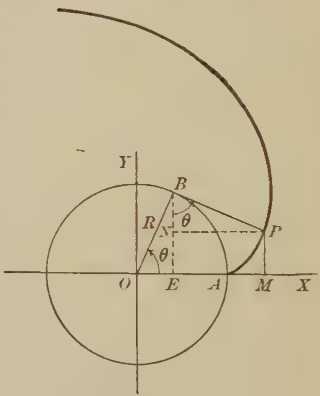
$$x = (R - r)\cos\theta + r\cos\frac{R-r}{r}\theta,$$

$$y = (R - r)\sin\theta - r\sin\frac{R-r}{r}\theta.$$

These equations can also be derived directly from the adjoining figure.

The Astroid. — In the hypocycloid for which $r = \frac{1}{4}R$, the rolling circle makes four revolutions in passing around the base, forming the astroid, or hypocycloid of four cusps.

129. The Involute of the Circle. — A string is wound about the circumference of a circle. One end is fastened to the circumference and the string is unwound. If the string is kept stretched the curve traced by the free end is called the *involute* of the circle.



If it begins to unwind at A, the arc AP of the involute is traced when the string has unwound as far as B, and the part unwound is BP.

BP is tangent to the circle at B, and

$$BP = \text{arc } BA = R\theta.$$

The equations are

$$x = R\cos\theta + R\theta\sin\theta,$$
$$y = R\sin\theta - R\theta\cos\theta.$$

NOTE. — The teeth of large-geared wheels are sometimes cut in the shape of epicycloids or hypocycloids to avoid the binding and friction that occur between straight teeth. Others are cut in the shape of involutes of a circle. In the latter type the parts having contact do not wear so rapidly, since the direction of motion at the point of contact is always perpendicular to the surface at that point.

PROBLEMS

1. Write the equations of the epicycloid in which $r = \frac{R}{3}$, make a table of values taking θ at intervals of 30°, and plot the curve.

2. Same as Problem 1 for the hypocycloid.

3. Make a table of values and plot the involute of the circle.

4. Show that the equations of the hypocycloid of four cusps can be reduced to $x = R\cos^3\theta$, $y = R\sin^3\theta$, which gives the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = R^{\frac{2}{3}}$.

5. Simplify the equations of the epicycloid and draw the curve when (*a*) $r = R$; (*b*) $r = \frac{R}{2}$.

6. Simplify the equations of the hypocycloid in which $r = \frac{R}{2}$ and draw the locus.

7. Derive the equations of the hypocycloid from a figure.

8. By using the definition of the curve draw the epicycloid for which

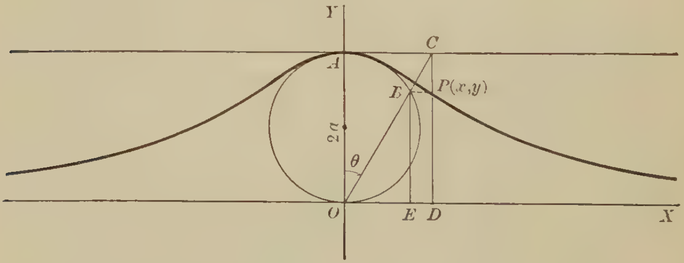
(*a*) $r = \frac{R}{5}$; (*b*) $r = \frac{2R}{3}$; (*c*) $r = \frac{R}{3}$; (*d*) $r = \frac{3R}{4}$.

9. Same as Problem 8 for the hypocycloid.

10. Show that if the origin is translated to the point $(R, 0)$ the equations of the epicycloid for which $r = R$ become $x' = 2R\cos\theta\,(1 - \cos\theta)$, $y' = 2R\sin\theta\,(1 - \cos\theta)$. Then transform these to polar coördinates and show that this particular epicycloid is a cardioid.

130. Algebraic Curves. — Algebraic curves of degree higher than the second have a great variety of forms and are best studied with the aid of the calculus. Two simple examples of these curves which we have encountered are the cubical parabola $y = ax^3$ and the semicubical parabola $y^2 = ax^3$. The majority of the curves in Chapter IX are found to be algebraic curves if their equations are transformed into rectangular form. Many of these curves are of historical importance, as the limaçon, the conchoid, the cissoid, and the lemniscate, given in Chapter IX, and the witch, the strophoid, and the ovals of Cassini, which will be discussed in the following sections.

131. The Witch. — A circle of radius a is drawn tangent to the x-axis through the origin, meeting the y-axis in A. The tangent to the circle at A is drawn, and from O a secant is drawn meeting the circle in B and the tangent in C. The locus of the projection of B upon the ordinate of C is called the *witch*.



To find its equation take the angle $\theta = AOC$ as a parameter. Then we have at once

$$x = OD = AC = 2\,a \tan\theta,$$

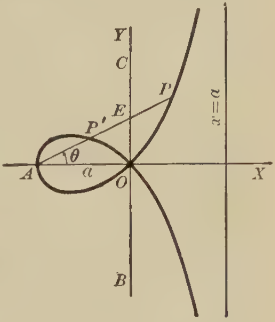
and $$y = DP = EB = OB\cos\theta = 2\,a\cos^2\theta.$$

Eliminating the parameter θ, we obtain the equation,

$$y = \frac{8a^3}{x^2 + 4a^2}.$$

The form of the equation shows that the curve is symmetrical with respect to the y-axis and has the x-axis as an asymptote.

132. The Strophoid. — The distance from a fixed point A to a fixed line BC is a. From A the perpendicular AO is drawn to BC and any other line AE is drawn meeting BC at E. On AE points P and P' are taken such that $P'E = EP = OE$. The locus of the points P and P' is a curve called the *strophoid.*



To find its equation take the fixed line as the y-axis and the x-axis through the fixed point. Let the coördinates of P and P' be (x, y) and (x', y'). Taking the angle OAE as a parameter θ, we have at once that

$$P'E = EP = OE = a \tan\theta.$$

Hence $\qquad x = EP\cos\theta = a\sin\theta,$

and $\qquad y = OE + EP\sin\theta = a\tan\theta\,(1 + \sin\theta).$

By eliminating the parameter and squaring both sides, we have the equation,

$$y^2 = x^2\frac{a + x}{a - x}.$$

To show that P' is on the same locus, we have

$$x' = -a\sin\theta,$$

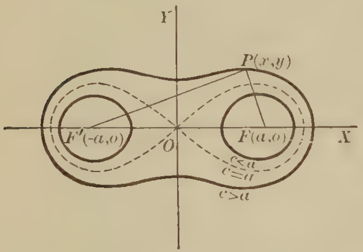
and $\qquad y' = a\tan\theta\,(1 - \sin\theta).$

Eliminating θ and rationalizing, we find that the coördinates of P' satisfy the given equation.

The curve is evidently symmetrical with respect to the x-axis and has the line $x = a$ as an asymptote.

133. The Ovals of Cassini. — The locus of the vertex of a triangle which has a constant base and has the product of its other two sides equal to a given constant is called the *ovals of Cassini.*

Call the length of the base $2a$, take the origin at its mid-point, and let the x-axis lie along the base. Then the conditions of the problem make $F'P \cdot FP$ equal to a constant, which we call c^2 since it is essentially positive.



Using the distance formula, we have

$$\sqrt{(x+a)^2+y^2} \cdot \sqrt{(x-a)^2+y^2} = c^2.$$

Simplifying this gives

$$(x^2+y^2+a^2)^2 - 4a^2x^2 = c^4.$$

Three types are possible, according to the relative values of c and a. If $c = a$, this last equation may easily be reduced to the form,

$$(x^2+y^2)^2 = 2a^2(x^2-y^2),$$

which becomes on transformation to polar coördinates,

$$\rho^2 = 2a^2 \cos 2\theta,$$

showing that in this case the curve is a lemniscate. (See page 168.)

PROBLEMS

1. Transform into rectangular form the polar equations of the following curves:

(*a*) the limaçon; (*d*) the cissoid;
(*b*) the cardioid; (*e*) the four-leafed rose;
(*c*) the conchoid; (*f*) the three-leafed rose.

2. Show that the following curves are not algebraic:

(*a*) the spiral of Archimedes;
(*b*) the lituus.

3. Plot the witch from its parametric equations.

4. Plot the strophoid.

5. Plot the ovals of Cassini for:

(*a*) $a = 4, c = 2.$ (*b*) $a = 2, c = 4.$

6. Find the polar equation of the strophoid.

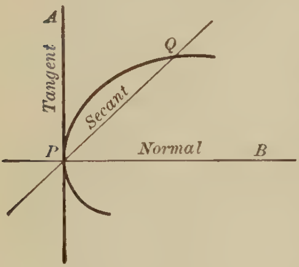
7. Transform the rectangular equation of the ovals of Cassini into polar form.

8. Derive from the figure the polar equation of the ovals of Cassini.

CHAPTER XI

TANGENTS AND NORMALS

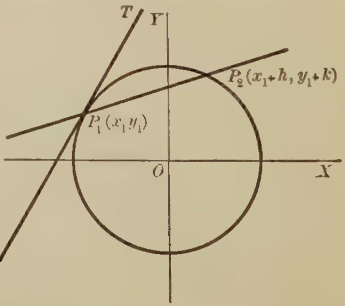
134. Definitions. — A line cutting a curve is called a *secant*. In the figure PQ is a secant. If Q is made to move along the curve towards P, it is clear that in general PQ will turn about P and approach a limiting position PA, which is called the position of tangency. Therefore



The tangent to a curve at a given point P is the limiting position of a secant through the points P and Q on the curve when Q approaches P.

The normal to a curve at a given point is the line perpendicular to the tangent at that point.

135. Slope of Tangent and Normal to a Circle at a Given Point. — To find the slope of a tangent to a curve at a given point $P_1(x_1, y_1)$, we first find the slope of the secant through P_1 and a neighboring point $P_2(x_1 + h, y_1 + k)$ on the curve. This will be $\frac{k}{h}$. As P_2 is made to approach P_1 as a limit, the secant approaches the tangent



and the slope of the tangent is the limiting value of $\frac{k}{h}$, which we write $\lim \frac{k}{h}$. Since both k and h approach the limit zero, special methods must be employed to find the value of the limit of their quotient.

In the case of the circle we proceed as follows. Since P_1 and P_2 are both on the circle, their coördinates satisfy the equation of the circle. Hence

$$x_1^2 + y_1^2 = r^2,$$
$$(x_1 + h)^2 + (y_1 + k)^2 = r^2.$$

Expanding and subtracting,

$$2\,hx_1 + h^2 + 2\,ky_1 + k^2 = 0,$$
$$h(2\,x_1 + h) = -k(2\,y_1 + k),$$

and
$$\frac{k}{h} = -\frac{2\,x_1 + h}{2\,y_1 + k}.$$

From this last relation, we have

$$\lim \frac{k}{h} = -\lim \frac{2\,x_1 + h}{2\,y_1 + k} = -\frac{x_1}{y_1}.$$

Therefore for the tangent to the circle $x^2 + y^2 = r^2$,

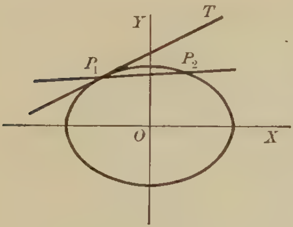
$$m = -\frac{x_1}{y_1}. \qquad (43)$$

As the normal is perpendicular to the tangent, its slope is the negative reciprocal of this expression, or $\frac{y_1}{x_1}$.

136. Slope of Tangent and Normal to an Ellipse at a Given Point. — The method of procedure is the same as that for the circle. Let the equation of the ellipse be

$$b^2x^2 + a^2y^2 = a^2b^2,$$

and the point of tangency be $P_1(x_1, y_1)$. A secant joining P_1 and a neighboring point $P_2(x_1 + h, y_1 + k)$ on the curve will have the slope $\frac{k}{h}\cdot$ Substituting the coördinates of P_1 and P_2 in the given equation,



$$b^2x_1{}^2 + a^2y_1{}^2 = a^2b^2,$$
$$b^2(x_1+h)^2 + a^2(y_1+k)^2 = a^2b^2.$$

Expanding and subtracting,

$$2\,b^2x_1h + b^2h^2 + 2\,a^2y_1k + a^2k^2 = 0,$$
$$h(2\,b^2x_1 + b^2h) = -k(2\,a^2y_1 + a^2k),$$
$$\frac{k}{h} = -\frac{2\,b^2x_1 + b^2h}{2\,a^2y_1 + a^2k}.$$

Hence $\lim\frac{k}{h} = -\lim\frac{2\,b^2x_1 + b^2h}{2\,a^2y_1 + a^2k} = -\frac{b^2x_1}{a^2y_1}.$

Thus for the tangent to the ellipse, the slope is

$$\boldsymbol{m = -\frac{b^2x_1}{a^2y_1}}. \tag{44}$$

The slope of the normal is $\frac{a^2y_1}{b^2x_1}.$

137. Slope of Tangent and Normal to the Parabola and Hyperbola at a Given Point. — By applying the method of §§ 135, 136 to the equations of the parabola and hyperbola, $y^2 = 2\,px$ and $b^2x^2 - a^2y^2 = a^2b^2$, we obtain the following results:

Tangent to parabola: $\boldsymbol{m = \frac{p}{y_1}}.$ **(45)**

Tangent to hyperbola: $\boldsymbol{m = \frac{b^2x_1}{a^2y_1}}.$ **(46)**

The slopes of the normals are: for the parabola, $-\dfrac{y_1}{p}$;

for the hyperbola, $-\dfrac{a^2y_1}{b^2x_1}$.

Exercise 1. Prove Formula 45.

Exercise 2. Prove Formula 46.

138. Equations of the Tangent and Normal at a Given Point. — To find the equation of the tangent or normal to a curve at a given point, it is necessary only to find the slope and then use the point-slope equation of the straight line,

$$y - y_1 = m(x - x_1).$$

Exercise 3. Show that the equation of the tangent to the circle $x^2 + y^2 = r^2$ at any point $P_1(x_1, y_1)$ is $x_1x + y_1y = r^2$.

Hint. — Since P_1 is on the circle, simplify by using $x_1^2 + y_1^2 = r^2$.

Exercise 4. Show that the equation of the tangent at any point $P_1(x_1, y_1)$ is:

(*a*) for the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\quad \dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$;

(*b*) for the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $\quad \dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1$;

(*c*) for the parabola $y^2 = 2px$, $\quad y_1y = p(x + x_1)$.

PROBLEMS

1. Find the slope of the tangent to each of the following curves for any point (x_1, y_1) on the curve:

(*a*) $xy = 2a^2$;

(*b*) $y = ax^3$;

(*c*) $(x - h)^2 = 2p(y - k)$;

(*d*) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$;

(*e*) $(x - h)^2 + (y - k)^2 = r^2$.

2. The same as Problem 1 for the following curves:

(*a*) $y^2 = ax^3$;

(*b*) $x^2 = 2py$;

(*c*) $x^2 - y^2 = a^2$;

(*d*) $(y - k)^2 = 2p(x - h)$;

(*e*) $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$.

3. In each of the following curves find the slope of the tangent and of the normal at the point indicated:

(*a*) $9x^2 + y^2 = 18, (1, -3)$;
(*b*) $x^2 + 4y^2 = 4, (2, 0)$;
(*c*) $x^2 - y^2 = 25, (10, 5\sqrt{3})$;
(*d*) $5x^2 - 12y^2 = 13, (\sqrt{5}, -1)$;
(*e*) $x^2 + y^2 = 169, (5, -12)$;
(*f*) $x^2 + y^2 = 25, (3, 4)$;
(*g*) $2y^2 = 5x, (10, 5)$;
(*h*) $x^2 = 36y, (12, 4)$.

4. Write the equation of the tangent and normal to each of the curves in Problem 3 at the point indicated.

5. Find the points at which the tangent to the corresponding curve in Problem 3 has the indicated slope:

(*a*) -3; (*b*) $-\frac{1}{2}\sqrt{3}$; (*c*) $-\frac{2}{3}\sqrt{3}$; (*d*) $\frac{\sqrt{5}}{12}$;
(*e*) $\frac{1}{2}$; (*f*) $\frac{1}{4}$; (*g*) $-\frac{1}{4}$; (*h*) $\frac{3}{2}$.

6. Prove that the tangents at the ends of the latus rectum of a parabola are perpendicular to each other.

7. Find the angle between each of the following pairs of curves. (By the angle between two curves is meant the angle between their tangents at the points of intersection.)

(*a*) $x^2 + y^2 = 25,$
$x + y = 7$;
(*b*) $x^2 + y^2 = 12,$
$y^2 = x$;
(*c*) $x^2 - y^2 = a^2,$
$xy = \sqrt{2}\,a^2$;
(*d*) $x^2 + 4y^2 = 4,$
$x + 2y = 2.$

8. Prove that the tangents at the ends of any chord through the focus of a parabola are perpendicular to each other.

9. Through what point of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ must a tangent and a normal be drawn so as to form with the principal axis an isosceles triangle?

10. An ellipse and hyperbola have the same foci. Prove that the tangents at their points of intersection are perpendicular to each other.

Hint. — The value of c is common to the curves.

11. Find the angle between the tangents at the ends of a latus rectum of

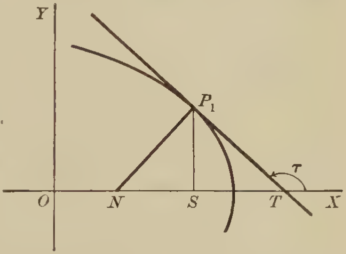
(*a*) the ellipse; (*b*) the hyperbola.

12. Prove that a circle described on a focal chord of a parabola as a diameter is tangent to the directrix.

13. Two perpendicular tangents with slopes m and $-\frac{1}{m}$ are drawn to the parabola $y^2 = 2px$. Find the distance between the horizontal lines drawn through the points of contact.

14. Find the slope of the focal chord joining the points of contact of the two tangents in Problem 13.

139. Tangent and Normal. Subtangent and Subnormal. — Let P_1T be the tangent and P_1N the normal to the curve in the figure at P_1. By the *length of the tangent* is meant the distance from the point of tangency to the point where the tangent meets the x-axis, *i.e.*, the length of P_1T. Similarly the length of the normal is the length of P_1N.



The *subtangent* is the projection of the tangent on the x-axis, *i.e.*, TS. The *subnormal* is the projection of the normal on the x-axis, *i.e.*, SN. The direction of reading the subtangent and subnormal is away from the intersection of the tangent with the x-axis; hence they are positive if they lie at the right of this point; negative if at the left.

To find the length of the subtangent, let m be the slope and τ the inclination of the tangent. Then

$$m = \tan \tau = \frac{SP_1}{TS} = \frac{y_1}{TS}.$$

Hence
$$TS = \frac{y_1}{m}. \qquad (47)$$

To find the length of the subnormal, observe that the slope of the normal is $-\frac{1}{m}$. Then

$$-\frac{1}{m} = \tan P_1NS = \frac{SP_1}{NS} = -\frac{SP_1}{SN}.$$

Therefore $$SN = my_1. \tag{48}$$

To find the lengths of the tangent and normal apply the right triangle theorem to the triangles SP_1T and NP_1S.

140. Tangent having a Given Slope. — Suppose that we desire to find the tangent to the curve $x^2 + 4y^2 = 8$ having the slope $\frac{1}{2}$.

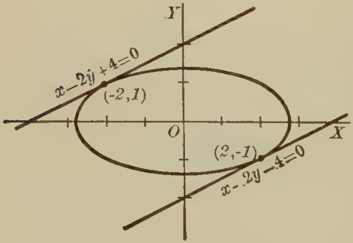
Here the slope is given and it is necessary to find the point of contact, which we call $P_1(x_1, y_1)$. Since P_1 is on the curve,

$$x_1^2 + 4y_1^2 = 8. \tag{a}$$

Now $x^2 + 4y^2 = 8$ is an ellipse, for which $a^2 = 8$ and $b^2 = 2$. Hence by (44) the slope of the tangent at P_1 is $-\frac{x_1}{4y_1}$. But by the conditions of the problem the slope is to be $\frac{1}{2}$. Hence

$$-\frac{x_1}{4y_1} = \frac{1}{2}. \tag{b}$$

Solving (*a*) and (*b*) simultaneously, the result is $x_1 = \mp 2, y_1 = \pm 1$. Thus there are two tangents with points of contact $(2, -1)$ and $(-2, 1)$ and equations



$$y \pm 1 = \tfrac{1}{2}(x \mp 2),$$

which reduce to

$$x - 2y \mp 4 = 0.$$

The method illustrated in the above problem is typical and may be applied to any problem of the same character.

141. Tangent from a Given External Point. — To find the equation of a tangent to a curve from a given external point we proceed as in the following problem.

Let the curve be $x^2 + 4y^2 = 8$ and the point $(1, \frac{3}{2})$. As before, let the point of contact be $P_1(x_1, y_1)$. Then we have

$$x_1^2 + 4y_1^2 = 8. \qquad (a)$$

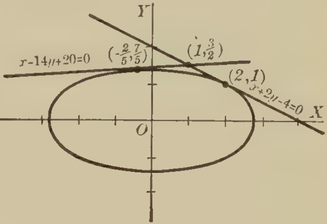
The slope of the tangent is, by (44), $-\dfrac{x_1}{4y_1}$. Hence the equation of the tangent is

$$y - y_1 = -\frac{x_1}{4y_1}(x - x_1).$$

Since this is to pass through $(1, \frac{3}{2})$ we have

$$\tfrac{3}{2} - y_1 = -\frac{x_1}{4y_1}(1 - x_1). \qquad (b)$$

Solving equations (a) and (b) simultaneously, we get $x_1 = 2$ or $-\frac{2}{5}$, $y_1 = 1$ or $\frac{7}{5}$.



Thus there are two tangents with points of contact $(2, 1)$, and $(-\frac{2}{5}, \frac{7}{5})$. For the first of these the slope is $-\frac{1}{2}$ and the equation

$$y - 1 = -\tfrac{1}{2}(x - 2),$$

or

$$x + 2y - 4 = 0.$$

For the second the slope is $\frac{1}{14}$ and the equation

$$y - \tfrac{7}{5} = \tfrac{1}{14}(x + \tfrac{2}{5}),$$

or

$$x - 14y + 20 = 0.$$

PROBLEMS

1. Write the equations of the tangent and normal to each of the following curves at the point indicated:

(*a*) $x^2 + y^2 = 13, (2, -3)$; (*e*) $16x^2 - 9y^2 = 112, (4, -4)$;
(*b*) $x^2 + 2y^2 = 18, (4, -1)$; (*f*) $x^2 - 8y = 0, (-4, 2)$;
(*c*) $y = 2x^3, (2, 16)$; (*g*) $x^2 - 4y^2 + 15 = 0, (1, 2)$;
(*d*) $xy = 24, (4, 6)$; (*h*) $3x^2 + y^2 = 28, (-3, 1)$.

2. The same as Problem 1 for the following:

(*a*) $x^2 - 2y^2 = 18, (-6, 3)$; (*e*) $x^2 + 4y^2 = 20, (2, 2)$;
(*b*) $2y^2 + 5x = 0, (-10, 5)$; (*f*) $xy + 18 = 0, (-3, 6)$;
(*c*) $y^2 = -2x^3, (-2, -4)$; (*g*) $y^2 - 4x^2 = 64, (3, -10)$;
(*d*) $y = 9x^2, (1, 9)$; (*h*) $7x^2 + 3y^2 = 55, (1, 4)$.

3. Find the length of the subtangent and subnormal for each curve in Problem 1.

4. Find the length of the subtangent and subnormal for each curve in Problem 2.

5. Find the equation of the tangent to each of the curves in Problem 1 parallel to the line $4x - 3y = 6$.

6. Find the equation of the tangent from the point $(5, -1)$ to each of the curves (*a*) and (*b*) in Problems 1 and 2.

7. Find the lengths of the subtangent, subnormal, tangent, and normal at any point (x_1, y_1) of

(*a*) $y^2 = 2px$; (*c*) $b^2x^2 - a^2y^2 = a^2b^2$;
(*b*) $b^2x^2 + a^2y^2 = a^2b^2$; (*d*) $x^2 + y^2 = r^2$.

8. Show that a line from the focus of the parabola $y^2 = 2px$ to the point where any tangent cuts the y-axis is perpendicular to the tangent.

9. Write and simplify the equation of the normal to each of the curves of Problem 7 at any point (x_1, y_1).

10. Prove that the tangents at the ends of the latus rectum of a parabola intersect on the directrix.

11. A tangent to the parabola $y^2 = 2px$ has the intercepts on the axes numerically equal. Find its equation.

12. Prove that the tangents to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the circle $x^2 + y^2 = a^2$ at points having the same abscissa meet on the x-axis.

13. At what points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are the tangents equally inclined to both axes?

14. Prove that the triangle formed by a tangent to the curve $2\,xy = a^2$, the x-axis, and a line joining the point of contact with the origin is isosceles.

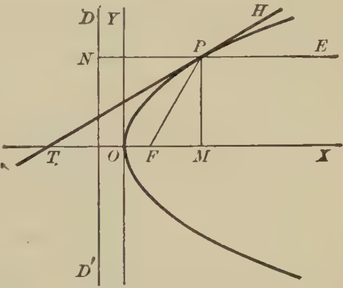
15. Find the area of the triangle formed by the tangent to the curve $2\,xy = a^2$ at any point and the coördinate axes.

16. If a tangent to a parabola meets the latus rectum produced at A and the directrix at B, prove that $FA = FB$, F being the focus.

17. If d and d' are the distances of the foci of an ellipse from a tangent, prove that dd' is the square of the semi-minor axes.

142. The Parabolic Reflector. — Let TP be tangent to the parabola $y^2 = 2\,px$ at $P(x, y)$, and let TM be the subtangent.

Applying the subtangent formula, $TM = 2\,x$; hence, $TO = OM = x$. Now if F is the focus, this gives



$$TF = x + \frac{p}{2}.$$

But $$FP = NP = x + \frac{p}{2}.$$

Hence, TPF is isosceles, *i.e., the tangent to a parabola makes equal angles with the focal radius and the principal axis.*

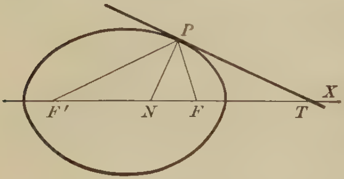
This principle is used in the construction of parabolic reflectors. These are reflectors whose surfaces are generated by revolving a parabola about its principal axis.

The lamp is placed at the focus. Then by the laws of optics a ray of light from F to any point on the surface P is reflected along PE so that angle TPF = angle HPE. But this makes angle HPE equal to angle PTF by the above proof. Therefore PE is parallel to OX. Thus all the rays of light from a lamp placed at F are reflected along lines parallel to the axis. (See § 61.)

Exercise 5. Use the above property of the tangent to devise a method of constructing a tangent to a given parabola at any point by ruler and compasses.

143. Angle between the Focal Radii and the Tangent to an Ellipse at any Point.

— Let PT be a tangent and PN a normal to the ellipse at any point $P(x, y)$ on the ellipse, whose foci are F and F'. We desire to show that these lines make equal angles with the focal radii to P.



The slope of $F'P$ is $\frac{y}{x+c}$; of FP, $\frac{y}{x-c}$; and of NP, $\frac{a^2y}{b^2x}$, by § 136. Hence

$$\tan F'PN = \frac{\dfrac{a^2y}{b^2x} - \dfrac{y}{x+c}}{1 + \dfrac{a^2y^2}{b^2x(x+c)}} = \frac{a^2xy + a^2cy - b^2xy}{b^2x^2 + b^2cx + a^2y^2}$$

$$= \frac{c^2xy + a^2cy}{a^2b^2 + b^2cx} = \frac{cy}{b^2};$$

$$\text{and } \tan FPN = \frac{\dfrac{y}{x-c} - \dfrac{a^2y}{b^2x}}{1 + \dfrac{a^2y^2}{b^2x(x-c)}} = \frac{b^2xy - a^2xy + a^2cy}{b^2x^2 - b^2cx + a^2y^2}$$

$$= \frac{a^2cy - c^2xy}{a^2b^2 - b^2cx} = \frac{cy}{b^2}.$$

Therefore $F'PN = FPN$; and the angles which the focal radii make with the tangent, being complementary to these, are also equal. (See § 71.)

Exercise 6. Show how to draw a tangent to an ellipse at any point.

144. Diameters of a Conic. — The locus of the middle points of a system of parallel chords of any conic is called a *diameter* of the conic.

Let a chord with slope m meet the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the points $P_1(x_1, y_1)$ and $P_2(x_1 + h, y_1 + k)$. Then by § 136 the slope of the chord is

$$m = \frac{k}{h} = -\frac{b^2(2x_1 + h)}{a^2(2y_1 + k)}.$$

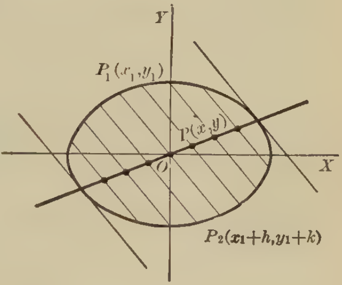
Now let $P(x, y)$ be the mid-point of the chord; then

$$2x = 2x_1 + h$$

and

$$2y = 2y_1 + k$$

by the mid-point formulas.



Substituting, we have, $m = -\dfrac{b^2x}{a^2y}$

or

$$y = -\frac{b^2}{a^2m}x, \tag{49}$$

which is the equation of the locus. From this equation we see that the diameter is a straight line through the center of the ellipse of slope $m' = -\frac{b^2}{a^2m}$. Transferring the m to the other side, we have an important relation between the slopes of a diameter and its chords,

$$mm' = -\frac{b^2}{a^2}. \tag{50}$$

In similar manner, we may show that the equation of the diameter of the parabola $y^2 = 2\,px$, is

$$y = \frac{p}{m}, \tag{51}$$

where m is the slope of the system of chords.

For the hyperbola we obtain as the equation of the diameter of a set of chords of slope m,

$$y = \frac{b^2}{a^2m}x. \tag{52}$$

The relation between the slopes of the chords and their diameter is

$$mm' = \frac{b^2}{a^2}. \tag{53}$$

Exercise 7. Prove Formula 51.

Exercise 8. Prove Formula 52.

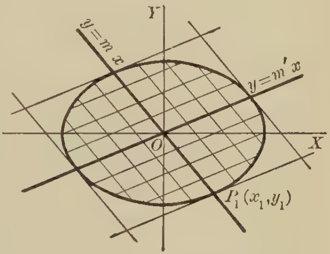
145. Conjugate Diameters of an Ellipse.

— The relation

$$mm' = -\frac{b^2}{a^2}$$

states the condition satisfied by m and m' if the line $y = m'x$ is the diameter of a set of chords of slope m. But as m and m' can be interchanged in (50) without altering the relation, we see that $y = mx$ is the diameter of the set of chords of slope m', parallel to the first diameter.

Two diameters which have slopes m and m' such that $mm' = -\frac{b^2}{a^2}$ are called *conjugate diameters*. Each diameter is one of the chords bisected by its conjugate.



It is readily seen that tangents at the ends of a diameter are parallel to the conjugate diameter.

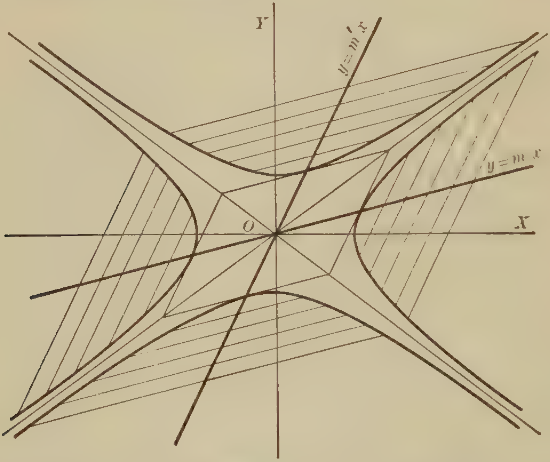
For, let $P_1(x_1, y_1)$ be the end of the diameter $y = mx$. The slope of the tangent at P_1 is $-\frac{b^2x_1}{a^2y_1}$ by § 136. But $\frac{x_1}{y_1} = \frac{1}{m}$, since P_1 lies on $y = mx$. Hence the slope of the tangent is $-\frac{b^2}{a^2m}$, which is also the slope of the conjugate diameter.

This property, together with Exercise 6, supplies a method of constructing a diameter conjugate to a given diameter with ruler and compasses.

146. Conjugate Diameters of the Hyperbola. — The relation between the slope of a diameter of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ and the slope of the corresponding chords has been found to be

$$mm' = \frac{b^2}{a^2}.$$

As the equation of the conjugate hyperbola, $-b^2x^2 + a^2y^2 = a^2b^2$, is obtained by substituting $-b^2$ for b^2 and a^2 for $-a^2$, it is easy to see that the corresponding relation for the diameter and chords of the conjugate hyperbola is $mm' = \frac{-b^2}{-a^2} = \frac{b^2}{a^2}$. That is, it is the same in both the hyperbola and its conjugate. As in the case of the ellipse, two diameters of slopes m and m' such that $mm' = \frac{b^2}{a^2}$ are called *conjugate diameters.*



Similarly, the chords bisected by the diameter $y = mx$ are parallel to the conjugate diameter $y = m'x$. Also the tangents to a hyperbola at the ends of a diameter are parallel to the conjugate diameter.

Exercise 9. Prove that the tangents at the ends of a diameter of a hyperbola are parallel to the conjugate diameter.

PROBLEMS

1. Find the diameter bisecting the system of chords of slope 2 in each of the following conics:

(a) $x^2 + 4y^2 = 16$;
(b) $x^2 - 9y^2 = 9$;
(c) $x^2 + y^2 = 13$;
(d) $2y^2 - 5x = 0$;
(e) $9x^2 + 16y^2 = 144$;
(f) $16x^2 - 9y^2 = 144$.

2. Find the diameters conjugate to those obtained in Problem 1.

3. What is the relation between the slopes of conjugate diameters of the ellipse $b^2y^2 + a^2x^2 = a^2b^2$?

4. Show that conjugate diameters of an ellipse lie in different quadrants.

5. What is the locus of the mid-points of a system of parallel chords with slope m, drawn in the parabola $y^2 = 2\,px$?

6. Show that conjugate diameters of a hyperbola lie in the same quadrant.

7. What relation must exist between a and b when two conjugate diameters meet the ellipse in the extremities of the latus rectum?

8. Prove that the major axis of an ellipse is greater than any other diameter.

Hint. — Show that $4\,a^2$ is greater than the square of any diameter.

9. Lines are drawn joining the ends of the major and minor axes of an ellipse. Show that the diameters parallel to these are conjugate.

10. Prove that the tangents to an ellipse at the ends of a diameter make equal angles with lines joining these points to a focus.

11. Tangents at the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ form a circumscribed parallelogram. Find its area.

Hint. — One conjugate diameter may be taken as a base, and the altitude may be found by using the normal form of the straight line equation.

12. Prove that the tangents to two conjugate hyperbolas at the ends of a pair of conjugate diameters meet on an asymptote.

13. Prove that if any pair of conjugate diameters of a hyperbola are equal, the hyperbola is equilateral.

14. Prove that in an equilateral hyperbola the asymptotes bisect the angle between any pair of conjugate diameters.

CHAPTER XII

GRAPHS OF FUNCTIONS AND EMPIRICAL EQUATIONS

147. Functions in General. — In Chapter II, § 21, the term *function* was defined as follows: *When two variables are connected by some law such that the value of one depends upon that of the other, the first variable is said to be a function of the other.* Various illustrations of this definition were given, but, in using this notion, we have hitherto dealt solely with the case that $y = f(x)$ represents the equation of some curve and $f(x)$ represents some mathematical expression involving the variable x. It is clear, however, from the definition that the notion of a function is capable of much wider applications and we shall now discuss such applications. The discussion will be limited to one-valued functions, *i.e.*, functions in which there is but one value of the dependent variable for each value of the independent variable.

The notation $y = f(x)$ means, according to its definition, that the value of y depends on that of x, or in other words that to each value assumed by x there is a corresponding value of y. The symbol $f(x)$ may represent some mathematical expression, but this is not necessary. For example, the student of elementary physics knows that if an object is dropped in a vacuum the formula $s = \frac{1}{2} gt^2$, where $g = 32.2$ approximately, expresses the distance (s feet) fallen in terms of the time (t sec.) since it was dropped. If the object falls in air, this formula no longer holds, but it is no less clear that s is a function of t. In like manner the horse-power (P) of a given type of gasoline motor is a function of the cubical contents (v) of its cylinders and we may write $P = f(v)$ without knowing any formula connecting P and v.

In the future, then, we shall regard y as a function of x, provided that to each value of x there corresponds a value of y, whether or not any formula is given.

148. General Problems. — Although the problems regarding functions are not primarily of a geometrical character, their study by means of the methods of analytic geometry is one of the most important applications of this subject. The problems to be considered are:

1. To study functions by means of their graphs;
2. To discover, if possible, analytic expressions for functional relations.

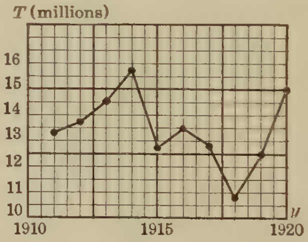
149. Plotting from Tabulated Data. — When one variable is a function of another, frequently the only information regarding this functional relation is a table of corresponding values of the variables, obtained by collecting statistics, by measurements, or by similar means. In plotting such a function the values of the variable which is regarded as independent are usually taken as abscissas and those of the dependent variable as ordinates.

The scales used should be as large as will permit of representing all the data on the paper. In general they will not be the same and they should be clearly marked on the axes. In choosing the scales the best practice is to have one space on the coördinate paper represent 1, 2, 5, 10, 20, 50, 100, etc., units of the variable concerned if its values are greater than 1; or 0.5, 0.2, 0.1, 0.05, 0.02, 0.01 0.005, etc., units if the values are less than 1. The reason for this is that with such scales it is usually possible to estimate an additional digit from the graph.

If the values of either variable are large, but their range is relatively small, we do not attempt to show the origin on the paper, but start from a value near the smallest to be exhibited. For example, if we had $y = f(x)$ and the values of y ranged from 102 to 116, it would be advisable to choose as the base line not the x-axis but the line $y = 100$. Then the point whose ordinate is 102 would be located two of the units chosen above this base line.

Example 1. — The amount of shipping entering New York Harbor is given in the adjoining table. Draw a graph illustrating this.

Year	Tonnage (000 omitted)
1911	13,400
1912	13,700
1913	14,500
1914	15,800
1915	12,600
1916	13,500
1917	12,900
1918	10,900
1919	12,500
1920	15,000



Solution. — If we let T represent the tonnage and y the number of the year, we have to draw the graph of T regarded as a function of y. Values of y will then be laid off on the horizontal axis. Since our paper is about 20 spaces wide, it is convenient to let 2 horizontal spaces represent 1 year and to start with $y = 1910$ at the intersection of the base lines.

The range of values of T is $15{,}800{,}000 - 10{,}900{,}000 =$ about 5,000,000. Letting 1 vertical space represent 500,000 tons, or 2 vertical spaces represent 1,000,000 tons, will require 10 spaces, slightly less than those available. We therefore use this scale and choose as the horizontal base line the line $T = 10{,}000{,}000$.

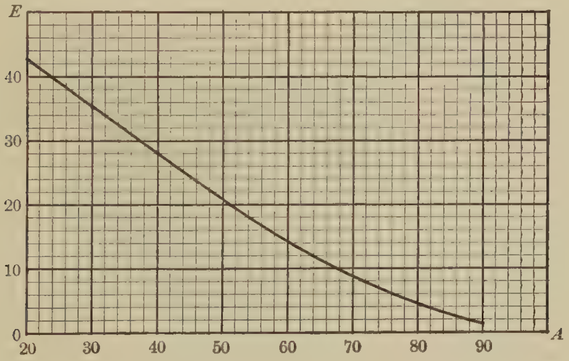
We now proceed to mark the scales and to plot the points. Values of y between those given evidently have no meaning, but we join adjacent points by straight lines to aid in carrying the eye from one point to another.

Example 2. — The following table gives the expectation of life at various ages. Illustrate by a graph.

Age	Expectation
20	42.20
30	35.33
40	28.18
50	20.91
60	14.10
70	8.48
80	4.39
90	1.42

Solution. — Let A represent age and E expectation; E is a function of A. If 1 space on the A-axis represents 2 years we need 35 spaces to cover the range of values of A. The same scale on the E-axis requires 21 spaces.

Proceeding as before, we plot the points. In this example values of A between those given have a meaning and there are corresponding values of E. We therefore join the points by as smooth a curve as possible, since it is reasonable to assume that the variation of E is regular.

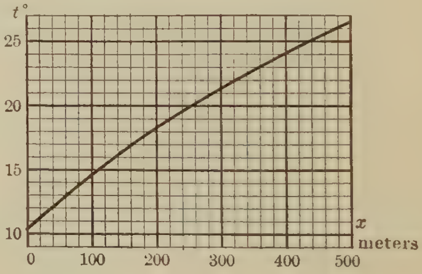


150. Plotting from Formulas. — When a functional relation between two variables is given by a formula, the method of procedure in plotting is the same as in the previous case except that the table of values must be calculated. If the variables are x and y and we wish to represent y as a function of x, we solve the equation for y in terms of x and proceed as in ordinary curve plotting with two exceptions. One is that we restrict the range of values of x to cover only those values which we wish to exhibit. The other is that values which have no concrete meaning in the problem are not plotted, even though they may happen to satisfy the equation.

Example. — The formula $t = 10.6 + 0.0415\,x - 0.0000193\,x^2$ gives the mean temperature t (°C) of a certain artesian well at various depths x (metres). Illustrate by a graph, letting x range up to 500 metres.

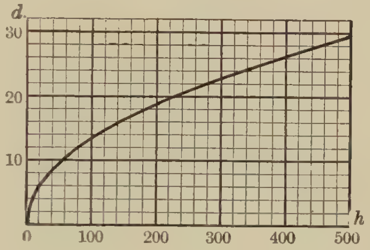
Solution. — Negative values of x naturally have no meaning. Let us calculate our table of values for x at intervals of 50. We let $t = 10$ be our horizontal base line and choose for our scales: 1 space = 20 metres; 1 space = 1°. We can then plot our values of t correct to 0.2°. The graph is a curve which is concave down. By inspection of the equation it is seen to be an arc of a parabola.

x	t
0	10.60
50	12.63
100	14.56
150	16.39
200	18.13
250	19.77
300	21.31
350	22.76
400	24.11
450	25.37
500	26.53



151. The Graph as an Aid in Computing. — One of the important uses of the graph of a function is the finding of additional pairs of values of the variables concerned. To illustrate this consider the following graph, which shows the distance (d miles) to the horizon at sea for various heights (h feet) above sea level. From the graph we can read off at once the value of h corresponding to a given value of d, and vice versa: *e.g.*, when $d = 20$, $h = 230$; when $d = 25$, $h = 360$; when $h = 120$, $d = 14.5$, etc.

h	d
0	0
50	9.3
100	13.2
150	16.2
200	18.7
250	20.9
300	22.9
350	24.7
400	26.4
450	28.0
500	29.5



This method is of practical value only when the graph is carefully drawn on a fairly large scale. It is most useful when a very great degree of accuracy in the results is not required and when the same graph can be used often enough to justify the labor of drawing it.

If a formula is given and the value of one variable corresponding to only one value of the other is desired, it is usually easier to calculate the desired value than to plot the graph. But when several values are required, the graphical method is much superior. If in the formula plotted in the preceding section we consider the problem of calculating the value of x for which $t = 15$, we can see how laborious such calculations can be even when the formulas are relatively simple.

When no formula is available the only alternative to using the graph is to interpolate by proportional parts.

This is not only more troublesome, but frequently less accurate than using the graph. The reason is that when we interpolate by proportional parts we assume that the portion of the graph between two adjacent plotted points is a straight line. That this is often far from being the case is easily seen from the above graph. For example, the graph shows that when $h = 20$, $d = 5.9$. If we interpolate to find d when $h = 20$, we have:

$$\frac{20}{50} = \frac{d}{9.3}\ .$$

or

$$d = \tfrac{2}{5} \times 9.3 = 3.72$$

In this case the interpolated result is too small.

PROBLEMS

1. Illustrate the following tables of statistics graphically:

(*a*) U. S. Merchant Marine

Year	Tonnage (000 omitted)
1911	7,640
1912	7,710
1913	7,890
1914	7,930
1915	8,390
1916	8,470
1917	8,870
1918	9,920
1919	12,910
1920	16,320

(*b*) U. S. Customs Average Rate

Year	Rate (%)
1860	19.6
1865	47.5
1870	47.1
1875	40.5
1880	43.7
1885	45.8
1890	44.5
1895	41.7
1900	49.7

(*c*) Annual Rainfall at New York

Year	Inches
1909	41.55
1910	35.98
1911	40.34
1912	38.50
1913	44.39
1914	33.50
1915	40.83
1916	33.17
1917	39.28
1918	33.59
1919	49.38

2. The premium ($\$P$) on a \$1000 life insurance policy for various ages (x yrs.) is given in the following table. Draw a graph exhibiting P as a function of x. Estimate from the graph the premium at age 32 and at age 43; also the age at which the premium is $52.

x	20	25	30	35	40	45	50	55	60
P	18.78	21.02	23.86	27.54	32.36	38.83	47.68	59.88	76.94

3. The following table gives the pressure of steam (P lbs.) at various temperatures (T° F.). Draw a graph showing P as a function of T. From the graph make a table showing the temperatures corresponding to pressures of 240, 250, 260, etc. up to 330 lbs.

T	10	20	30	40	50	60	70	80	90	100
P	240.0	259.2	274.3	286.9	297.8	307.4	316.0	323.9	331.1	337.8

4. The following table gives the pressure per square inch (p lbs.) of saturated steam corresponding to the volume (v cu. ft.) per pound. Draw a graph showing p as a function of v. Estimate from the graph the values of p corresponding to $v = 20$ and $v = 10$; also the values of v corresponding to $p = 20$ and $p = 40$.

v	53.9	26.4	14.0	7.0	4.3	2.7
p	6.9	14.7	28.8	60.4	101.9	163.3

5. An object is thrown vertically upward with an initial velocity of 96 feet per second. Its velocity after t seconds is given by the formula $v = 96 - 32\,t$, where v is measured in feet per second. Draw the graph of this function.

6. The velocity (v ft. per sec.) of an object thrown directly upwards is given by the formula $v^2 = u^2 - 64\,s$, where u (ft. per sec.) is the initial velocity and s (ft.) is the height reached. Plot the graph of v as a function of s when (a) $u = 32$; (b) $u = 64$.

7. The volume (V) of a sphere in terms of the diameter (D) is given by the formula, $V = \frac{1}{6}\pi D^3$. Plot the graph, letting D range from 0 to 10. Estimate from the graph the correct diameters for $V = 100, 200, 300, 400, 500$.

8. The pressure per square inch (p lbs.) upon a parachute falling in air in terms of its velocity (v ft. per sec.) is approximated by the formula $p = 0.00303\ v^2$. Plot the graph, letting v range from 0 to 30.

9. The melting point (θ° C.) of an alloy of lead and zinc containing $x\%$ of lead is given by the formula $\theta = 141.4 + 0.620\ x + 0.0130\ x^2$. Draw the graph, letting x vary from 40% to 90%. For what percentage is the melting point 200°?

10. The barometric pressure (p in.) in terms of the height (h ft.) is given by the formula $p = 30\ e^{-0.000023\ h}$. Plot the graph, letting h vary from 0 to 30,000. What are the values of h corresponding to $p = 28$ and $p = 20$?

11. The number of years (n) required for a sum of money at compound interest at the rate i (%) to double itself is given approximately by the formula $n = \frac{69}{i} + \frac{1}{3}$. Plot the graph, letting i vary from 1 to 10. Estimate from the graph the values of i corresponding to integral values of n from 10 to 20.

12. Two burners are 10 feet apart. At a point on the line joining them the amount of heat (H calories per second) received from the burners is given by the formula $H = \frac{60}{x^2} + \frac{480}{(10 - x)^2}$, where x is the distance of the point from one of the burners. Plot the graph of this function and determine from the graph the value of x which makes H least.

13. In a circle of radius R the area of a segment cut off by a chord subtending an angle x° at the center is given by the formula $A = \frac{R^2}{2}\left(\frac{\pi x}{180} - \sin x\right)$. Plot the graph of this function when (*a*) $R = 10$; (*b*) $R = 20$. From the graph find the value of the angle when the area of the segment is one-third the area of the circle.

152. Derivation of Equations. — No general rules can be given for deriving the equation expressing a functional relation, since the procedure depends upon the laws of the subject from which the problem is taken. In order to express y as a function of x we must obtain from the conditions of the problem an equation of the form $y = f(x)$, in which $f(x)$ contains no variable other than x. We shall illustrate by giving three important types of problems which are commonly met in applications of mathematics.

153. A Geometrical Problem. — A rectangle is inscribed in a semi-circle of fixed radius r. It is required to express the area of the rectangle as a function of the base.

Solution. — Let A be the area and let $x = EB$ be the base. If we denote the altitude BC by y, we have at once

$$A = xy.$$

But A is to be expressed as a function of x. Hence we must find a relation connecting x and y so that y can be expressed in terms of x.

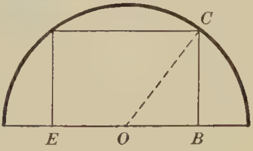


FIG. 1.

Let O be the center of the circle and draw the radius OC.

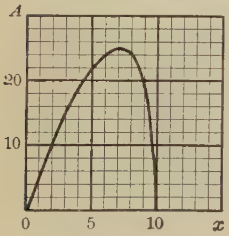


FIG. 2.

Then $$r^2 = \overline{OB}^2 + \overline{BC}^2;$$

that is $$r^2 = \frac{x^2}{4} + y^2,$$

or $$y = \tfrac{1}{2}\sqrt{4r^2 - x^2}.$$

Substituting this above gives the result

$$A = \frac{x}{2}\sqrt{4r^2 - x^2}.$$

The graph of this function is plotted in Fig. 2.

154. Linear Functions. — These occur when one variable changes with respect to the other at a constant rate. By the rate of change of y with respect to x is meant the change in y per unit change in x; that is, if x_1, y_1 and x_2, y_2 are two pairs of values, the rate of change is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

We now proceed to prove two theorems analogous to those of § 37.

Theorem I. *If $y = f(x)$ is linear, the rate of change of y with respect to x is constant.*

Proof. By hypothesis, $y = f(x)$ is of the form $y = b + mx$, where b and m are constants. If x_1, y_1 and x_2, y_2 are any two pairs of values of x and y,

$$y_1 = b + mx_1$$

and

$$y_2 = b + mx_2.$$

Subtracting, we have

$$y_1 - y_2 = m(x_1 - x_2),$$

whence

$$\frac{y_1 - y_2}{x_1 - x_2} = m.$$

The left member of this is the rate of change of y with respect to x. As it equals m, a constant, the theorem is proved.

Theorem II. *If y changes at a constant rate with respect to x, $y = f(x)$ is linear.*

Proof. Let m denote the rate of change and x_1, y_1 be a fixed pair of values of the variables. Since the rate of change is constant, we have for *any* other pair of values x, y:

$$\frac{y - y_1}{x - x_1} = m,$$

whence

$$y - y_1 = m(x - x_1).$$

This is a linear equation.

Corollary. If the rate of change of y with respect to x is m and $y = b$ when $x = 0$, $y = b + mx$.

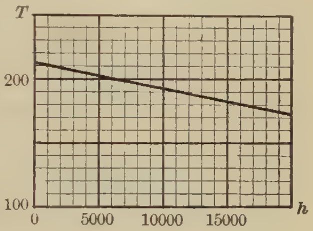
From the above it is apparent that the graph of a function changing at a constant rate is a straight line whose slope is the rate of change (due allowance being made for the use of different scales in plotting). Hence we need but two pairs of values to plot the graph and we can use any of the forms of Chapter III in writing the equation.

Example. — If T is the boiling point of water and h is the height above sea-level, T is a linear function of h. $T = 212°$ F. when $h = 0$, and $T = 202°$ F. when $h = 5000$ ft. Express T as a function of h, find the rate of change of T, and draw the graph.

Solution. — As we have two pairs of values, we use the two-point form of the linear equation, obtaining

$$\frac{T-212}{h-0} = \frac{212-202}{0-5000} = -0.002$$

or $$T = 212 - 0.002\,h.$$



By inspection the rate of change is $-0.002°$ per foot, the minus sign showing that the boiling point falls as h increases.

The graph is the straight line joining the two given points.

155. Variation, Direct and Inverse. — The phrases *varies as* and *varies inversely as* frequently occur in stating various laws. The statement "y varies as x" (or "y is proportional to x") means simply that $y = kx$, where k is a constant. The statement "y varies inversely as x" (or "y is inversely proportional to x") means that $y = \frac{k}{x}$, where k is a constant. Obviously the value of k can be determined if one pair of values of the variables is given.

Example. — The square of the time of revolution (T) of a planet about the sun varies as the cube of its distance (x) from the sun. The distance of the earth from the sun is 92.9 millions of miles. Express T as a function of x.

Solution. — The statement of the problem gives

$$T^2 = kx^3.$$

But when $x = 92.9$, $T = 1$ year.
Hence

$$1 = (92.9)^3 k \quad \text{or} \quad k = \frac{1}{(92.9)^3}.$$

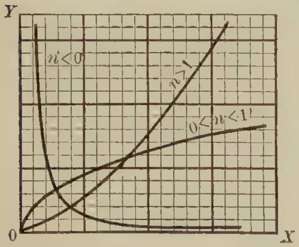
Substitution of this gives

$$T = \frac{x^{\frac{3}{2}}}{(92.9)^{\frac{3}{2}}},$$

whence $$T = 0.001117\, x^{\frac{3}{2}}.$$

The graph is evidently a semi-cubical parabola.

156. Several Common Algebraic Functions. — Besides linear functions we frequently have functions of the form $y = kx^n$ and $y = a + bx + cx^2$. In the first quadrant, if k is positive, the graph always assumes one of the forms given in the figure. If n is a fraction $\pm\frac{p}{q}$, where p and q have no common factor, it is easy to see by considerations of symmetry that the remainder of the graph is in the second quadrant when p is even, in the fourth quadrant when q is even, and in the third quadrant when p and q are both odd.



The graph of $y = a + bx + cx^2$ is a parabola with its axis perpendicular to the x-axis. For this function the greatest or the least value of y can be found by determining the vertex as directed in Chapter VIII. For example, if $y = 2 + 8x - 3x^2$, we complete the square in x and obtain $y = -3(x - \frac{4}{3})^2 + \frac{22}{3}$. Hence the vertex is at

$x = \frac{4}{3}$, $y = \frac{22}{3}$. As the square term is negative $\frac{22}{3}$ is the *greatest* value of y.

PROBLEMS

1. Express the area of an equilateral triangle as a function of the length of a side and plot the graph.

2. Express the area of a regular hexagon as a function of the length of a side and plot the graph.

3. A plane distant a units from the center of a sphere of radius r cuts off a segment of volume $V = \frac{\pi}{3}(a^3 - 3\,ar^2 + 2\,r^3)$. If x denotes the greatest thickness of the segment, express V as a function of x.

4. If the perimeter of a rectangle is 100, express the area as a function of the length. Find the length for which the area is greatest.

5. If the area of a rectangle is 100, express the perimeter as a function of the length. Plot the graph and estimate the length which gives the least perimeter.

6. It is desired to fence off three sides of a rectangular plot of ground, a stone wall being available for the fourth side. If 120 yards of fence material are available, express the area of the rectangle as a function of the length. Find the dimensions giving the largest rectangle.

7. In the triangle ABC, the angle C is a right angle, $AC = 20$, and $BC = 10$. A rectangle is drawn having one angle coinciding with C and the vertex opposite to C lying on AB. Express the area of the rectangle as a function of the length of the side lying along AC, draw the graph, and determine the length which makes the area largest.

8. In a triangle ABC the length of the base AB is 120 ft. and the length of the corresponding altitude is 60 ft. A rectangle is drawn having one side lying along AB and the other vertices on AC and BC. Express the area of the rectangle as a function of its length, draw the graph, and find the area of the largest possible rectangle of this kind.

9. A right circular cylinder is inscribed in a right circular cone of radius r and height h. Express the height of the cylinder as a function of its radius. Plot the graph of this function.

10. A rectangular box of constant volume C has a square base. Express the total surface of the box as a function of the length of one side of the base. Draw the graph of the function.

11. A rectangular box has square ends and the sum of the length and the girth is 84 in. Express the volume as a function of the length of one side of the square end. Plot the graph and estimate the dimensions which will give the largest volume. (Such a box is the largest that can be sent by parcels post.)

12. Express the distance D from the point (10, 0) to any point P on the parabola $y^2 = 16x$ as a function of the abscissa of P. Plot $D = f(x)$ and estimate from the graph the value of x which makes D least.

13. The formula $V = V_0\left(1 + \frac{T}{273}\right)$ gives the volume of expanding gas as a function of the temperature T, where the pressure remains constant and V_0 denotes the volume when $T = 0°$. Plot the graph of this function for the case that $V_0 =$ 100 cu. ft. What is the rate of change in the volume?

14. The telegraph rate over a certain distance is 40 cts. for 10 words or less, and 3 cts. for each additional word. Write a formula giving the cost of a message (C cts.) as a function of the number of words (n). For what values of n is this formula valid?

15. A steel rail expands at a constant rate with respect to the temperature. If its length is 30 ft. at a temperature of 0° C., and 30.00163 at a temperature of 50° C., express the length as a function of the temperature. Draw a graph on such scales as will make it possible to read off the length for any temperature between 0° and 50°.

16. The length (l in.) of a wire stretched by a force (p lbs.) increases at a constant rate as the tension increases. If $l = 34.5$ when $p = 100$ and $l = 35.3$ when $p = 500$, at what rate does the length increase? Write l as a function of p.

17. On ascending a mountain it is found that the boiling point of water falls at the rate of 2° per thousand feet. If the boiling point ($T°$) is 212° at sea-level, express T as a function of the height (h ft.).

18. Express the amount (\$$A$) of a note for \$100 at simple interest 6% as a function of the time (n yrs.). At what rate is A increasing?

19. Plot the graphs of the following:

(*a*) $y = \frac{1}{4} x^{\frac{4}{3}}$; (*c*) $y = 10 x^{-2}$; (*e*) $y = 2 x^{-3}$;
(*b*) $y = \frac{1}{2} x^{\frac{3}{2}}$; (*d*) $y = .015 x^3$; (*f*) $y = 6 x^{-\frac{1}{2}}$.

20. Sketch the graphs of the following, taking all arbitrary constants positive:

(*a*) $y = kx^n$, n integral;
(*b*) $y = kx^{-n}$, n integral;
(*c*) $y = kx^{\frac{p}{q}}$, p odd, q even;
(*d*) $y = kx^{-\frac{p}{q}}$, p odd, q even;
(*e*) $y = kx^{\frac{p}{q}}$, p even, q odd;
(*f*) $y = kx^{-\frac{p}{q}}$, p even, q odd;
(*g*) $y = kx^{\frac{p}{q}}$, p odd, q odd;
(*h*) $y = kx^{-\frac{p}{q}}$, p odd, q odd.

21. The number (n) of vibrations per second of a string varies inversely as the length (l in.). Express n as a function of l. Sketch the graph. How could the value of the arbitrary constant be found?

22. The velocity of a falling body (v ft. per sec.) varies as the square root of the distance (s ft.) through which it has fallen.

If $v = 40$ when $s = 25$, express v as a function of s. What is the graph?

23. The areas of regular polygons of the same number of sides vary as the square of the length of a side. By careful measurement the area of a regular pentagon is found to be 27.53 sq. in. and the length of a side is 4 in. Write a formula for the area as a function of the length of a side.

24. The acceleration due to gravity (g ft. per sec. per sec.) varies inversely as the square of the distance (d miles) from the center of the earth. If $g = 32$ on the surface of the earth and the radius of the earth is taken as 4000 miles write the formula expressing g as a function of d. Sketch the graph.

25. The consumption of coal by a certain locomotive varies as the square of the velocity; when the velocity is 16 miles per hour the coal consumption is 2 tons per hour. Write a formula giving the coal consumption per hour for any velocity. Draw the graph.

26. If other expenses of running the locomotive in Problem 25 were \$5 per hour and coal cost \$10 per ton, express the cost of making a run of 100 miles as a function of the velocity. Plot the graph and estimate what velocity would be most economical.

27. The pressure (p) of a gas varies inversely as the volume (v). If $p = 50$ lbs. when $v = 3$ cu. ft., express p as a function of v and plot the graph.

157. Empirical Formulas. — Suppose that n pairs of values of the variables x and y are known and it is desired to express y in the form $y = f(x)$. The problem stated is clearly insoluble. For, if we take *any* equation involving n arbitrary constants and substitute the values of x and y, we get n simultaneous equations which can in general be solved. Thus we seem to have an infinite number of possible solutions of the above problem. On the other hand there is no reason to believe that any one of these possible equations would be satisfied by any pair of values of the variables other than those given.

However, the given values of x and y usually represent measurements of some kind and are themselves not exact. If then we can find a simple formula *nearly* satisfied by the measured values, this may be regarded as a solution of the problem stated above. It will serve as a practical means of calculating approximations to other values of the variables. Its form may suggest hitherto unknown laws connecting the variables, which can be established by later investigations.

Such formulas are called *empirical*, in contradistinction to *rational* formulas or formulas derived by reasoning. The formulas of pure mathematics are rational. Those of science are usually empirical in their application. The discovery of empirical formulas is sometimes called *curve fitting*. The aim in each case is to get a formula such that the calculated values of the dependent variable differ from the measured or observed values by less than the errors due to unavoidable imperfections in measurement.

158. General Procedure. — In finding an empirical formula the work falls into three parts: 1, choosing the type of equation; 2, finding the best values of the arbitrary constants; 3, calculating the deviations between the observed values of the dependent variable and those given by the formula.

The choice of the type of equation is sometimes dictated by theoretical considerations. If there is no reason for preferring a particular type, the choice is a matter of intelligent guess-work and experiment. The first thing to do is always to plot the data and draw a graph. Its shape will often indicate the proper equation. Other methods will be indicated later.

The second step must always be carried out even if we know the form of the equation in advance. For example, if we know that the equation is of the form $y = mx$, we cannot find m by substituting any pair of values, since different pairs will give different values. This is due to the impossibility of measuring x and y exactly. As an experiment draw a straight line joining the origin to the point (16, 13), read off from the graph the values of y for integral values of x and see how many exactly satisfy the equation $y = \frac{13}{16}x$.

The purpose of the last step is to check the work. As stated before the desired formula should be such that the differences between the calculated and observed values of y will be less than the errors made in measuring the latter. If this is not attainable, the formula must give results accurate enough at least for the purposes for which it is used.

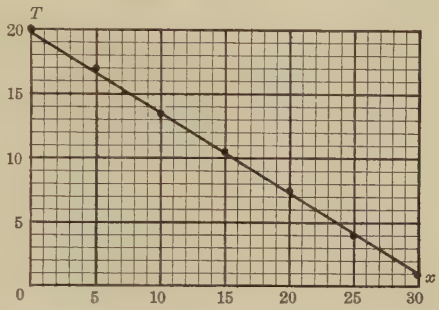
In the following we shall confine the discussion to problems where the formulas are of the common types $y = b + mx$, $y = a + bx + cx^2$, $y = kx^n$, and $y = ke^{nx}$.

159. The Type $y = b + mx$. — Since a large number of functions change at a constant rate, we shall first consider fitting linear graphs. The method is best explained by an example.

In the following table T is a function of x and the values of T have been measured to the nearest 0.5 unit. Plotting the points, we see that they lie nearly in a straight line and therefore conclude that a linear function will give a "good fit."

Setting $T = b + mx$, we now have to determine the proper values of the arbitrary constants b and m. There are several ways* of doing this; we shall give two. First of all, we can draw the line which seems to come nearest to all the plotted points and read off the values of b and m from the graph. In the above figure $b = 19.8$ apparently. To get the slope take two points on the graph near the ends and use the slope formula. Thus, when $x = 6$, $T = 16$, and when $x = 27$, $T = 3$. Hence $m = -\frac{13}{21} = -0.62$. The required formula is

x	T
0	20.0
5	17.0
10	13.5
15	10.5
20	7.5
25	4.0
30	1.0



$$T = 19.8 - 0.62\,x.$$

To get good results by this method we need a large sheet of plotting paper.

The second method is what is called the "method of averages." If we substitute the values of x and T in the equation $T = b + mx$, we obtain the following equations:

$$\begin{aligned} 20.0 &= b \\ 17.0 &= b + 5\,m \\ 13.5 &= b + 10\,m \\ 10.5 &= b + 15\,m \\ 7.5 &= b + 20\,m \\ 4.0 &= b + 25\,m \\ 1.0 &= b + 30\,m \end{aligned}$$

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* The best method of obtaining the coefficients is by the method of least squares. This involves principles beyond the scope of this book and will not be discussed. For ordinary purposes the methods given here are satisfactory.

We divide these equations into two nearly equal groups, say the first four in one group and the last three in the other, and add. This gives two equations

$$4b + 30m = 61$$
$$3b + 75m = 12.5$$

which can be solved simultaneously. Solving, we find $b = 20$ and $m = -0.633$. Hence the required formula is

$$T = 20.0 - 0.633x.$$

The values of T as calculated by this formula are given in the adjoining table under the heading T_c. Subtracting these from the observed values T_0, we find the residuals $T_0 - T_c$. As none of the latter are greater than 0.2 and the observed values all had a possible error of 0.25 (since they were measured only to the nearest $\frac{1}{2}$ unit) we conclude that our formula is a satisfactory one.

x	T_0	T_c	$T_0 - T_c$
0	20.0	20.0	0
5	17.0	16.8	+0.2
10	13.5	13.7	−0.2
15	10.5	10.5	0
20	7.5	7.3	+0.2
25	4.0	4.2	−0.2
30	1.0	1.0	0

160. Principle Involved in the Method of Averages. — If we assume a formula of the type $T = b + mx$ and calculate values for b and m as in the previous section, we find that the substitution of a particular value for x in $b + mx$ will not give the observed value T_0 of T in general. Thus for any value of x we have $T_0 - (b + mx)$ not equal to zero but equal to the residual $T_0 - T_c$. In the example referred to $20 - b$, $17 - (b + 5m)$, $13.5 - (b + 10m)$, and $10.5 - (b + 15m)$ are the residuals for $x = 0, 5, 10$, and 15. If we set the sum of these equal to zero, we have

$$61 - (4b + 30m) = 0,$$

which is the same as the first simultaneous equation of the previous section

$$4b + 30m = 61.$$

Thus the method of averages is merely a calculation of values of the arbitrary constants of the assumed formula which will make the algebraic sum of the residuals equal to zero for each of two groups of values of the variables. This makes the average residual zero, and we assume that the values of the constants thus obtained give a satisfactory formula.

The reason for using *two* groups is that we have two arbitrary constants. If we had only one, we would sum all the equations; if we had three, we would divide the data into three groups.

161. First Differences. — If in a table we subtract a value of one variable from the next following, we call the result a *first difference*. First differences are indicated by prefixing Δ to the letter representing the variable. In the adjoining table every value of Δx is $+5$, and the values of ΔT are tabulated opposite the corresponding values of x and T. There is of course no ΔT for the last value of x in the table.

In general, if y is a linear function of x, and therefore changes at a constant rate, it is clear that for equal values of Δx the corresponding values of Δy must be equal, and conversely. For Δy is the change in y due to a change of Δx units in x, and therefore equals Δx times the rate of change of y with respect to x.

x	T	ΔT
0	20.0	−3.0
5	17.0	−3.5
10	13.5	−3.0
15	10.5	−3.0
20	7.5	−3.5
25	4.0	−3.0
30	1.0	

This fact affords a method for deciding upon the availability of a linear formula without plotting the data. If the values of the independent variable are given at equal intervals, calculate the first differences for the dependent variable. If these are approximately equal, a linear relation may be assumed. In the given example the values of x are given at intervals of 5 units and ΔT is approximately constant; hence it is reasonable to assume a linear relation.

PROBLEMS

In each of the following find a linear formula approximately representing the second variable as a function of the first. (*a*) by graphical means; (*b*) by the method of averages.

1.

x	y
25	0.58
40	0.72
50	0.82
60	0.89
75	1.01
80	1.04
90	1.10

2.

x	y
20	110.35
38	110.71
62	111.18
74	111.43
92	111.77

3.

t	d
24.7	0.00791
37.1	0.00808
49.3	0.00820
61.7	0.00834
66.8	0.00845
71.5	0.00849

4.

x	y
19	76.00
25	77.80
30	79.35
36	81.10
40	82.35
45	83.70
50	85.40

5.

L	T
2.2	1.92
3.2	2.22
3.9	2.45
4.5	2.65
5.0	2.82
5.5	2.97
5.9	3.11
6.3	3.25

6.

x	y
5	31.0
10	28.5
15	26.1
20	23.5
25	20.8
30	18.3
35	15.8
40	13.4

7.

x	y
3.6	47.2
5.2	52.2
5.9	54.7
6.7	57.0
9.6	66.1
12.2	74.2

8.

x	y
3	100.32
6	100.65
10	101.07
12	101.30
15	101.61

9. In one of the above problems calculate the values of the dependent variable from the empirical formula and find the residuals.

10. In one of the above sets of data show the availability of a linear formula by finding the first differences.

11. The latent heat of steam (L calories) is shown in the following table:

T	70	85	100	115	130
L	557	546	536	526	515

for various temperatures ($T°$). Find an empirical formula.

12. The force (F lbs.) which would just lift a weight (w lbs.) was found in experiments with a crane to be as follows. Deduce an empirical formula.

w	100	200	300	400	500	600	700	800
F	8.5	12.8	17.0	21.4	25.6	29.9	34.2	38.5

13. The length of an iron bar (L cm.) at various temperatures ($T°$) was found to be as follows. Find an empirical formula.

T	20	30	40	50	60	70
L	1000.00	1000.20	1000.38	1000.57	1000.75	1000.95

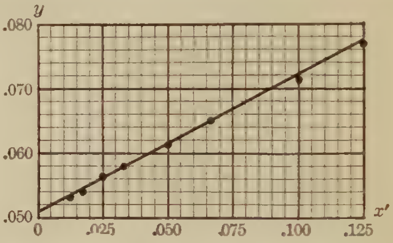
162. Equations of the forms $y = a + bx^2$ and $y = a + \frac{b}{x}$. These forms involve two arbitrary constants and can therefore be obtained in the same way as linear equations. Since the first is the equation of a parabola symmetrical with respect to the y-axis, it is not applicable unless y has its least (greatest) value when $x = 0$ and increases (decreases) indefinitely with x. The graph of $y = a + \frac{b}{x}$ is an equilateral hyperbola with the asymptote $y = a$ and will sometimes fit a set of values in which the value of the dependent variable appears to approach a constant.

If in the first case we set $x' = x^2$, the equation $y = a + bx^2$ becomes $y = a + bx'$, which is linear. Hence, if we plot (x', y) and the plotted points lie nearly in a straight line, we may assume that $y = a + bx^2$ is the proper type. The values of a and b can then be approximated from the figure or obtained by the method of averages. Similar remarks apply to the type $y = a + \frac{b}{x}$.

Example. — Find an empirical formula fitting the data in the following table.

x	y	$x' = \frac{1}{x}$
8	.0770	.1250
10	.0715	.1000
15	.0650	.0667
20	.0614	.0500
30	.0582	.0333
40	.0563	.0250
60	.0540	.0167
80	.0532	.0125

Solution. — Inspection of the table shows that as x increases y decreases, and indicates that y approaches a definite limit. It seems reasonable therefore to try a formula of the type $y = a + \frac{b}{x}$. We set $x' = \frac{1}{x}$ and calculate the values of x'. Plotting the points (x', y), we find that they lie nearly in a straight line, whose equation is found from the graph to be $y = 0.051 + 0.21\ x'$. Hence an approximate formula is $y = 0.051 + \dfrac{0.21}{x}$.



To obtain better values of a and b we divide the data into two equal groups and use the method of averages. Substituting the values of x' and y in the equation $y = a + bx'$, we have:

$$\begin{array}{rcll} .0770 &=& a + .1250\,b & \\ .0715 &=& a + .1000\,b & \\ .0650 &=& a + .0667\,b & \\ .0614 &=& a + .0500\,b & \\ \hline .2749 &=& 4\,a + .3417\,b & (1) \end{array} \qquad \begin{array}{rcll} .0582 &=& a + .0333\,b & \\ .0563 &=& a + .0250\,b & \\ .0540 &=& a + .0167\,b & \\ .0532 &=& a + .0125\,b & \\ \hline .2217 &=& 4\,a + .0875\,b & (2) \end{array}$$

Solving equations (1) and (2) simultaneously, we find that $b = 0.209$ and $a = 0.0509$. Hence $y = 0.0509 + 0.209\,x'$, or $y = 0.0509 + \dfrac{0.209}{x}$.

If we calculate the values of y by this formula, we find that the greatest residual is numerically 0.0004. However, we can obtain nearly as good results by using the simpler formula $y = 0.051 + \dfrac{0.21}{x}$.

163. The Parabola $y = a + bx + cx^2$. — If it is thought that the values of x and y may satisfy an equation of this type, the coefficients a, b, and c can be obtained by the method of averages as in the case of linear formulas. Here the given values must be divided into three groups, since there are three arbitrary constants.

As the work of finding these constants is much longer than in the linear case, it is desirable to have some quick method of determining the suitability of this type in advance. If the values of x are equidistant, this is easily found. In fact, we have the following theorem.

If $y = a + bx + cx^2$ and Δx is a constant, Δy is a linear function of x.

Proof. Let $\Delta x = h$. Then for any value of x:

$$\begin{aligned} y &= a + bx + cx^2 \\ y + \Delta y &= a + b(x + h) + c(x + h)^2 \\ &= a + bx + bh + cx^2 + 2\,chx + ch^2. \end{aligned}$$

Hence

$$\begin{aligned} \Delta y &= bh + 2\,chx + ch^2 \\ &= (bh + ch^2) + 2\,chx, \end{aligned}$$

which is a linear function of x.

The converse of this is readily established. To determine whether or not Δy is linear we apply § 161. The difference between two successive first differences is called a *second difference* and is denoted by $\Delta^2 y$. If Δy is linear, § 161 tells us that $\Delta^2 y$, the first difference of Δy, must be constant. Hence we have the corollary:

If $y = a + bx + cx^2$ and Δx is constant, $\Delta^2 y$ is constant, and conversely.

From these results we conclude that if the values of x are equidistant and those of $\Delta^2 y$ are approximately equal, the proper formula is of the type $y = a + bx + cx^2$.

Example. — Find an empirical equation for the following table of values of t and T, regarding the former as the independent variable.

t	T	ΔT	$\Delta^2 T$	T_c
0	15.1	4.9	−0.5	15.2
5	20.0	4.4	−0.4	20.0
10	24.4	4.0	−0.4	24.4
15	28.4	3.6	−0.4	28.4
20	32.0	3.2	−0.3	32.0
25	35.2	2.9		35.3
30	38.1			38.1

Solution. — Here $\Delta t = 5$. Calculating ΔT and $\Delta^2 T$, we find that the latter is nearly constant and therefore assume a relation of the type

$$T = a + bt + ct^2.$$

Dividing the data into three groups and substituting for t and T, we obtain the following equations.

$$\begin{aligned}
15.1 &= a \\
20.0 &= a + 5\,b + 25\,c \\
24.4 &= a + 10\,b + 100\,c \\
\hline
59.5 &= 3\,a + 15\,b + 125\,c \qquad (1)\\
28.4 &= a + 15\,b + 225\,c \\
32.0 &= a + 20\,b + 400\,c \\
\hline
60.4 &= 2\,a + 35\,b + 625\,c \qquad (2)\\
35.2 &= a + 25\,b + 625\,c \\
38.1 &= a + 30\,b + 900\,c \\
\hline
73.3 &= 2\,a + 55\,b + 1525\,c \qquad (3)
\end{aligned}$$

Subtracting equation (2) from equation (3) and dividing by 20, we get

$$b + 45\,c = 0.645. \tag{4}$$

Multiplying equation (1) by 2 and equation (2) by 3, subtracting, and dividing the result by 25, we get

$$3\,b + 65\,c = 2.488. \tag{5}$$

Now eliminate b from equations (4) and (5). This gives

$$c = -0.0079.$$

Hence $b = 1.00^+$. Substituting $b = 1$ and $c = -0.0079$ in equation (1) gives finally $a = 15.2$.

Our formula is then

$$T = 15.2 + t - 0.0079$$

The calculated values of $T(T_c)$ are seen to agree almost exactly with the given values.

PROBLEMS

Find a formula of the type $y = a + bx^2$ approximately satisfied by the following sets of values.

1.

x	y
0.1	1.64
0.2	1.63
0.3	1.60
0.4	1.58
0.5	1.56
0.6	1.52
0.7	1.49
0.8	1.43

2.

x	y
5.4	0.0493
6.9	0.0821
8.5	0.123
9.6	0.154
10.9	0.197
11.8	0.234
12.8	0.274
14.1	0.328

3.

x	y
0	54.4
4.5	64.6
6.3	74.6
7.7	84.5
8.9	94.4

4.

x	y
5.2	19.9
5.7	22.0
6.3	24.7
7.4	30.1
8.2	35.0

Find a formula of the form $y = a + \frac{b}{x}$ approximately satisfied by the following sets of values.

5.

x	y
2	4.2
4	8.1
6	9.6
8	10.1
10	10.6
12	10.8

6.

x	y
1.2	26.7
1.5	22.7
1.7	20.6
2.0	18.5
2.5	16.1
3.0	14.3
3.5	13.1
4.0	12.3

7.

x	y
320	0.0468
240	0.0459
180	0.0444
140	0.0431
120	0.0420
100	0.0407
80	0.0382
60	0.0345

8.

x	y
36.8	16.6
31.5	17.0
26.3	17.2
21.0	17.4
15.8	18.2
12.6	18.7

Find a formula of the form $y = a + bx + cx^2$ approximately satisfied by the following sets of values.

9.

x	y
0	1.818
2	1.822
3	1.826
4	1.832
5	1.840
6	1.850

10.

x	y
0	0.26
1	0.60
2	0.98
3	1.39
4	1.82
5	2.29
6	2.77
7	3.29
8	3.84

11.

x	y
1.5	36
2.0	52
2.5	72
3.0	92
3.5	115
4.0	140
4.5	168
5.0	199
6.0	267

12. The time (n yrs.) required for money to double itself at compound interest is given in the following table for various rates (i%). Express n as a function of i.

i	1	2	3	4	5	6	7	8	9	10
n	69.4	35.0	23.5	17.7	14.2	11.9	10.2	9.0	8.0	7.3

13. The following table gives the average weight (W lbs.) for men of various heights (h in.). Express W as a function of h.

h	64	65	66	67	68	69	70	71	72	73	74
W	136	140	144	148	152	156	161	166	172	178	184

14. The following table gives the time of flight (t sec.) of a certain projectile for various ranges (R thousand yards). Express t as a function of R.

R	1	2	3	4	5	6
t	2.1	4.7	7.8	11.2	15.1	19.4

15. The following table shows the velocity (v ft./sec.) of a stream at various depths (x tenths of the total depth). Express v as a function of x.

x	0	1	2	3	4	5	6	7	8
v	4.27	4.86	5.14	5.15	4.85	4.24	3.36	2.16	0.67

164. Equations of the form $y = kx^n$. — Here k and n are unknown constants, whose values are to be determined. This type of equation can be reduced at once to the linear form by taking the logarithm of both members. This gives $\log y = \log k + n \log x$. Setting $y' = \log y$, $k' = \log k$, and $x' = \log x$, we have $y' = k' + nx'$, an ordinary linear equation.

Hence, to fit a formula of this type to a set of values of x and y, we first tabulate the values of $x' = \log x$ and $y' = \log y$. We then plot the points (x', y') and, if these appear to lie on a line, we can then determine k' and n either graphically or by the method of averages.

Example. — Find a formula approximately satisfied by the following table of values.

x	y	$x' = \log x$	$y' = \log y$	y_c
10	1.06	1.0000	0.0253	1.06
20	1.33	1.3010	0.1239	1.33
30	1.52	1.4771	0.1818	1.52
40	1.68	1.6021	0.2253	1.67
50	1.81	1.6990	0.2577	1.80
60	1.91	1.7782	0.2810	1.91
70	2.01	1.8451	0.3032	2.02
80	2.11	1.9031	0.3243	2.11

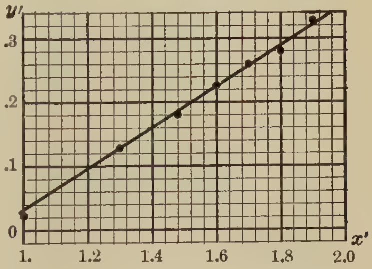
Solution. — Since the values of x are evenly spaced, we can easily show by calculating Δy and $\Delta^2 y$ that the desired formula is neither linear nor parabolic. Nor do the values of y appear to approach a definite limit.

We therefore look up the logarithms of x and y and tabulate them. Plotting the points (x', y'), we find that they lie nearly in a line. Hence the formula $y = kx^n$ is applicable and we begin with the linear relation $y' = k' + nx'$.

Proceeding by the method of averages, we have the following equations.

$$\begin{aligned}
0.0253 &= k' + 1.0000\,n \\
0.1239 &= k' + 1.3010\,n \\
0.1818 &= k' + 1.4771\,n \\
0.2253 &= k' + 1.6021\,n \\
\hline
0.5563 &= 4\,k' + 5.3802\,n \quad (1)
\end{aligned}$$

$$\begin{aligned}
0.2577 &= k' + 1.6990\,n \\
0.2810 &= k' + 1.7782\,n \\
0.3032 &= k' + 1.8451\,n \\
0.3243 &= k' + 1.9031\,n \\
\hline
1.1662 &= 4\,k' + 7.2254\,n \quad (2)
\end{aligned}$$



Solving equations (1) and (2) simultaneously, we obtain $k' = -0.3055$ and $n = 0.3305$. Thus $y' = -0.3055 + 0.3305\,x'$.

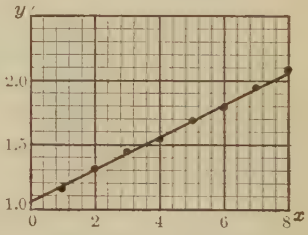
Since $k' = -0.3055$, $\log k = -0.3055 = 9.6945 - 10$, and $k = 0.4949$. Hence $y = 0.4949\ x^{0.3305}$. The calculated values of y are given in the table.

165. Equations of the form $y = ke^{nx}$. — As stated in § 114 many variables are connected by an exponential law of this kind. Since $\log_{10} e = 0.4343$ or $e = 10^{0.4343}$, the equation $y = ke^{nx}$ is equivalent to $y = k(10)^{n'x}$, where $n' = 0.4343\,n$. Taking logarithms of both sides, we have $\log y = \log k + n'x$. Setting $y' = \log y$ and $k' = \log k$, we have $y' = k' + n'x$.

Hence a formula of this type is indicated if the points (x, y'), where $y' = \log y$, lie approximately in a straight line. The method of finding k' and n' is the same as in the previous section.

Example. — Find a formula approximately satisfied by the following table of values.

x	y	$y' = \log y$	y_c
1	15.3	1.1847	15.3
2	20.5	1.3118	20.5
3	27.4	1.4378	27.4
4	36.6	1.5635	36.7
5	49.1	1.6911	49.1
6	65.6	1.8169	65.6
7	87.8	1.9435	87.8
8	117.6	2.0704	117.5



Solution. — It is easy to show that the proper formula is neither linear nor parabolic. Tabulating the values of $y' = \log y$ and plotting the points (x, y'), we find that these points lie nearly in a straight line. Hence the desired formula is of the form $y = ke^{nx}$, or $y = k\,(10)^{n'x}$.

Substituting the values of x and y' in the corresponding linear relation, $y' = k' + n'x$, we have

$$\begin{array}{rcl} 1.1847 &=& k' + n' \\ 1.3118 &=& k' + 2\,n' \\ 1.4378 &=& k' + 3\,n' \\ 1.5635 &=& k' + 4\,n' \\ \hline 5.4978 &=& 4\,k' + 10\,n' \quad (1) \end{array} \qquad \begin{array}{rcl} 1.6911 &=& k' + 5\,n' \\ 1.8169 &=& k' + 6\,n' \\ 1.9435 &=& k' + 7\,n' \\ 2.0704 &=& k' + 8\,n' \\ \hline 7.5219 &=& 4\,k' + 26\,n' \quad (2) \end{array}$$

Solving equations (1) and (2) simultaneously, we get $n' = 0.1265$ and $k' = 1.0582$. Hence $y' = 1.0582 + 0.1265\,x$.

Since $k' = \log k = 1.0582$, $k = 11.43$. Thus $y = 11.43\,(10)^{0.1265\,x}$. Using the relation $10 = e^{2.3026}$, we transform this at once into

$$y = 11.43\,e^{0.2913x}.$$

166. The foregoing sections constitute only an introduction to the subject of empirical equations. Only a few of the simpler algebraic and exponential forms have been used and nothing has been given regarding periodic functions. For a more extensive treatment special textbooks* on the subject should be consulted.

PROBLEMS

Find formulas of the type $y = kx^n$ approximately satisfied by the following sets of values.

1.

x	y
12	2.29
13	2.35
14	2.41
15	2.47
16	2.52
17	2.57

2.

x	y
2	0.49
8	1.13
25	2.25
50	3.43
100	5.21
150	6.59
200	7.73

3.

x	y
750	0.00220
1000	0.00197
1500	0.00171
2000	0.00157
2500	0.00147
3000	0.00140

Find formulas of the type $y = ke^{nx}$ approximately satisfied by the following sets of values.

4.

x	y
4	10.5
10	11.4
16	12.3
22	13.4
28	14.6
34	15.6
54	20.3
62	22.7

5.

x	y
0	7.50
44	7.25
138	6.75
238	6.25
347	5.75
530	5.00

6.

x	y
0.0	9.95
6.9	9.45
21.7	8.45
38.6	7.45
57.6	6.45
80.2	5.45

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* For example, Merriman's *Method of Least Squares* and Lipka's *Graphical and Mechanical Computation.*

7. The following table gives the number (N) of bacteria in a certain culture at various times (t hrs.). Express N as a function of t. ($N = ke^{nt}$.)

t	0	1	2	3	4	5	6	7
N	200	330	544	898	1478	2440	4020	6620

8. The dip of the sea horizon (D'') at various heights (h ft.) is given in the following table. Express D as a function of h. ($D = kh^n$.)

h	10	20	30	40	50	60
D	186	263	322	372	416	455

9. The following table gives the intercollegiate track records for various distances (t sec., d yds.). Express t as a function of d. ($t = kd^n$.)

d	100	220	440	880	1760	3520
t	9.8	21.2	48.8	116.0	257.8	657.6

10. The following table gives the difference between the temperature of a heated body and that of the air ($T°$) for various times (t min.). Express T as a function of t. ($T = ke^{nt}$.)

t	0	1	2	3	4	5	6	7
T	64.1	59.1	54.2	49.9	46.4	43.5	41.1	39.1

11. The following table gives the time of revolution about the sun (T yrs.) and the radius of the orbit (R million miles) for each of the planets. Express T as a function of R. ($T = kR^n$.)

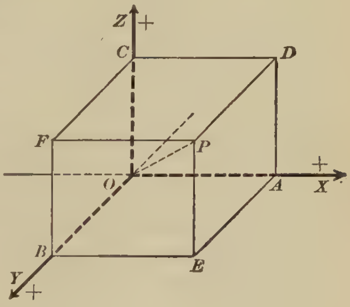
R	36.0	67.2	92.9	141.5	483.3	886.0	1781.9	2791.6
T	0.32	0.62	1.00	1.88	11.86	29.46	84.02	164.78

CHAPTER XIII

SOLID ANALYTIC GEOMETRY

167. Plane Analytic Geometry deals with the properties of figures which are wholly contained within a single plane. In Solid Analytic Geometry this restriction is removed. The effect of the addition of one dimension is to generalize the work of the previous chapters, the number of variables being increased to three, the loci of equations being surfaces, etc. The student should note carefully the correspondences between the two systems.

168. Coördinates. — The position of a point is determined with reference to three mutually perpendicular planes, XOY, YOZ, and XOZ. These are called the *coördinate planes*, and designated as the xy-, yz-, and xz-*planes* respectively. Their lines of intersection, OX, OY, and OZ, are called *coördinate axes* and designated as the x-, y-, and z-*axes* respectively; and their common point, O, is called the *origin.*



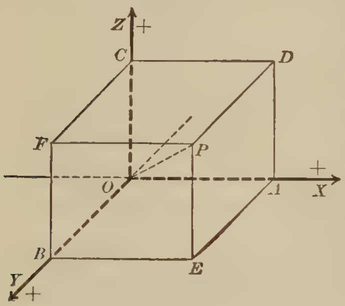
The *coördinates* of a point P are its distances from the respective coördinate planes measured parallel to the coördinate axes. The *x-coördinate* is the distance from the yz-plane parallel to the x-axis, the *y-coördinate* that from the xz-plane parallel to the y-axis, and the *z-coördinate* that from the xy-plane parallel to the z-axis.

For the point P in the figure,

$$\begin{aligned} x &= FP = BE = CD = OA, \\ y &= DP = AE = CF = OB, \\ z &= EP = BF = AD = OC. \end{aligned}$$

Corresponding to each point there are evidently always three coördinates. Conversely, a point is completely located by its coördinates. For the x-coördinate fixes it in a plane parallel to the yz-plane (or perpendicular to the x-axis), and similarly for the other coördinates; and these three planes meet in but one point. In plotting a point, we usually take

$$x = OA,\ y = AE,\ z = EP.$$



The positive directions are as indicated in the figure. The coördinate planes divide space into eight *octants*, which are distinguished by the signs of the coördinates of the points within them. For example the octant containing P in the figure has all the coördinates positive in sign.

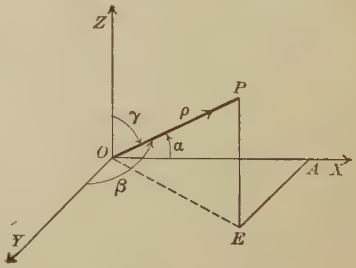
169. Radius Vector and Direction Cosines. — *The distance of a point from the origin is called its radius vector, and is denoted by* ρ.

For the point P in the figure $\rho = OP$. It is evident that

$$\begin{aligned} \rho^2 &= \overline{OE}^2 + \overline{EP}^2 \\ &= \overline{OA}^2 + \overline{AE}^2 + \overline{EP}^2. \end{aligned}$$

Hence

$$\boldsymbol{\rho^2 = x^2 + y^2 + z^2}. \tag{54}$$



The angles between the line OP and the positive halves of the x-, y- and z-axes are called the *direction angles* of the line OP and are denoted by α, β, and γ, respectively. The cosines of these angles are called the *direction cosines* of the line. They evidently fix its direction. From the figure it is readily seen that

$$x = \rho \cos \alpha, \qquad y = \rho \cos \beta, \qquad z = \rho \cos \gamma. \tag{55}$$

Formulas 54 and 55 give at once the important relation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \tag{56}$$

It is sometimes convenient to locate a point by means of its radius vector and its direction angles. In this case we write

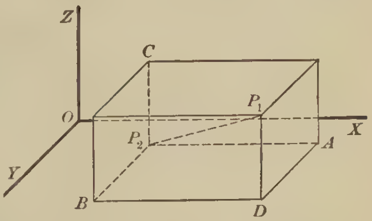
$$P(x, y, z) \equiv P(\rho, \alpha, \beta, \gamma).$$

This mode of representation is analogous to the polar coördinate system in plane analytic geometry.

170. Distance between Two Points. — *The distance between any two points* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *is given by the formula,*

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}. \tag{57}$$

To prove this, pass planes through the given points P_1 and P_2, parallel to the coördinate planes. These will form a rectangular parallelopiped, of which P_1P_2 is the diagonal.



By elementary geometry,

$$\overline{P_1P_2}^2 = \overline{P_2A}^2 + \overline{AD}^2 + \overline{DP_1}^2.$$

But $P_2A = x_1 - x_2$, $AD = y_1 - y_2$, and $DP_1 = z_1 - z_2$.

Hence we have the formula.

171. Direction of a Line. — *The direction angles of a line not passing through the origin are defined as the direction angles of a parallel line through the origin with the same positive direction.*

The positive direction on a line is arbitrary and indicated by the order in which its end-points are read. It is evident that if the direction is changed, the direction angles are replaced by their supplements. Thus, if the direction angles of P_2P_1 are α, β, and γ, those of P_1P_2 are $\pi - \alpha$, $\pi - \beta$, and $\pi - \gamma$. In all cases the direction angles are in value between 0 and π.

In the above figure the edges of the parallelopiped are parallel to the coördinate axes; hence the direction angles of the line P_2P_1 are:

$$\alpha = AP_2P_1, \qquad \beta = BP_2P_1, \qquad \gamma = CP_2P_1.$$

Then we have at once

$$\cos\alpha = \frac{x_1 - x_2}{d}, \quad \cos\beta = \frac{y_1 - y_2}{d}, \quad \cos\gamma = \frac{z_1 - z_2}{d}. \qquad (58)$$

Since the direction cosines are connected by Formula 56, they may be found if three numbers to which they are proportional are known. For if

$$\cos\alpha : \cos\beta : \cos\gamma = a : b : c,$$

then

$$\frac{\cos^2\alpha}{a^2} = \frac{\cos^2\beta}{b^2} = \frac{\cos^2\gamma}{c^2} = \frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}{a^2 + b^2 + c^2}$$
$$= \frac{1}{a^2 + b^2 + c^2},$$

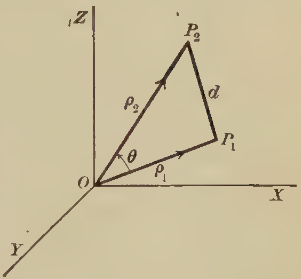
whence

$$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \qquad \cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$
$$\cos\gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

Any three numbers proportional to the direction cosines of a line are called *direction numbers* of the line.

172. Angle between Two Lines. — *The angle between two lines which do not meet is defined as the angle between two intersecting lines parallel to the given lines and having the same positive directions.* If the lines are parallel, the angle between them is 0 or π, according to their directions.

We now derive a formula for the angle between two lines in terms of their direction cosines. Let the lines through the origin parallel to the given lines be OP_1 and OP_2, where the coördinates of P_1 and P_2 are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Let d be the distance between these points, let ρ_1 and ρ_2 be their radii vectores, and let θ be the angle between the lines. Then



$$\cos\theta = \frac{\rho_1^2 + \rho_2^2 - d^2}{2\,\rho_1\rho_2}.$$

But

$$\begin{aligned}\rho_1^2 &= x_1^2 + y_1^2 + z_1^2,\\ \rho_2^2 &= x_2^2 + y_2^2 + z_2^2,\\ d^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.\end{aligned}$$

$$\therefore \cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\rho_1\rho_2}.$$

Now formula (55) gives $\frac{x_1}{\rho_1} = \cos\alpha_1$, etc. Hence this last equation becomes

$$\boldsymbol{\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2.} \quad \textbf{(59)}$$

When the two lines are parallel, we have

$$\alpha_1 = \alpha_2, \qquad \beta_1 = \beta_2, \qquad \text{and} \quad \gamma_1 = \gamma_2,$$

or
$$\alpha_1 = \pi - \alpha_2, \quad \beta_1 = \pi - \beta_2, \quad \text{and} \quad \gamma_1 = \pi - \gamma_2.$$

When the two lines are perpendicular, $\theta = \frac{\pi}{2}$ and (59) becomes

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0. \tag{60}$$

PROBLEMS

1. Find the radius vector and its direction cosines for each of the following points:

(*a*) $(3, 4, 5), (6, 0, 0), (-6, 1, 3)$;
(*b*) $(-2, 3, -1), (7, -4, 5), (1, 0, 1)$;
(*c*) $(0, 0, 2), (0, 3, -4), (-2, -2, 1)$.

2. Plot each of the above points.

3. Find the angles of the triangles whose vertices are given in Problem 1.

4. Generalize the definition of symmetry with respect to a line so that it will apply to symmetry with respect to a plane. Show that the point (x, y, z) is symmetrical to $(-x, y, z)$ with respect to the yz-plane; to $(x, -y, z)$ with respect to the xz-plane; to $(x, y, -z)$ with respect to the xy-plane.

5. Show that the point (x, y, z) is symmetrical to $(-x, -y, z)$ with respect to the z-axis; to $(-x, y, -z)$ with respect to the y-axis; to $(x, -y, -z)$ with respect to the x-axis.

6. Show by two methods that the following are the vertices of a right triangle:

(*a*) $(-2, 3, 5), (-5, -3, 3), (0, 0, 11)$;
(*b*) $(2, 6, -5), (-1, 0, -7), (4, 3, 1)$.

7. Find the area of each of the above triangles.

8. Show that the following points are vertices of a regular tetrahedron:

(*a*) (4, 0, 0), (0, 4, 0), (0, 0, 4), (4, 4, 4);
(*b*) (6, −2, 1), (3, 1, 1), (3, −2, 4), (2, −3, 0).

9. Find the coördinates of the point determined as follows:

$$\rho = 8, \qquad \cos\alpha = \tfrac{1}{2}, \qquad \cos\gamma = -\tfrac{1}{3}.$$

10. Show that the coördinates of the point P_0 dividing the line P_1P_2 in the ratio $r_1 : r_2$ are

$$x_0 = \frac{r_2x_1 + r_1x_2}{r_1 + r_2}, \qquad y_0 = \frac{r_2y_1 + r_1y_2}{r_1 + r_2}, \qquad z_0 = \frac{r_2z_1 + r_1z_2}{r_1 + r_2}.$$

11. What is the locus of points for which

(*a*) $z = 0$;
(*b*) $y = k$;
(*c*) $y = x = 0$;
(*d*) $x = y$;
(*e*) $x = 3, z = -2$;
(*f*) $\rho = 6$;
(*g*) $\cos\alpha = 0$;
(*h*) $\cos\alpha = \cos\beta = 0$?

12. Show that the three points (1, 4, − 2), (4, 7, 1), and (−2, 1, −5) are in a straight line.

13. Find the lengths of the medians of the triangle whose vertices are (3, 2, 1), (1, −4, 3), and (−1, 0, −5).

14. Prove that the points (4, 2, 1), (0, 0, 0), (2, 3, −1), and (6, 5, 0) form a parallelogram.

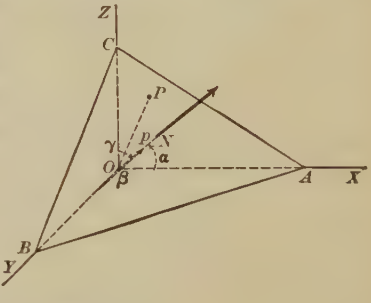
15. What is the projection of $P(-2, 3, -4)$ on
(*a*) the *xy*-plane;
(*b*) the *z*-axis?

173. The Locus in Solid Geometry. — *The locus of a point in space satisfying a given condition is in general a surface.* To illustrate, consider the locus of a point at a given distance *a* from a given fixed point *C*. This is evidently the definition of the surface of a sphere of radius *a* and center *C*. Again the locus of a point at a constant distance *a* from a given straight line *l* is evidently a circular cylindrical surface having *l* as an axis and *a* as a radius.

Extending the definition of the equation of a locus given in Chapter II, this shows that the locus of a single equation in the three variables, x, y, and z, is in general a surface.

If the coördinates of a point satisfy *two* equations simultaneously, its locus must be common to the two surfaces which are the loci of the equations, *i.e.*, it must be their intersection, which is a curve or set of curves. Finally the points determined by a set of *three* equations will be the intersections of the surface determined by the third with the curves determined by the first two, and hence will be isolated points. Note the analogy to plane analytic geometry, where *one* equation determines a curve, and *two* equations the isolated points in which the curves meet.

174. The Normal Equation of the Plane. — The perpendicular upon a plane from the origin is known as the *normal axis* and the distance of the plane from the origin is known as the *normal intercept.* A plane is completely determined if the length of the normal intercept p, and the direction angles, α, β, and γ, of the normal axis are known.



To find the normal equation of a plane, take any point $P(x, y, z)$ in the plane and join it to the origin. Draw the normal axis ON. Let ρ_1 be the radius vector of P, and α_1, β_1, γ_1, its direction angles; let α, β, γ be the direction angles of ON; and let θ be the angle between OP and ON.

Then by Formula 59,

$$\cos\theta = \cos\alpha_1\cos\alpha + \cos\beta_1\cos\beta + \cos\gamma_1\cos\gamma.$$

But $\cos\theta = \dfrac{ON}{OP} = \dfrac{p}{\rho_1}$

Eliminating θ between these equations,

$$\rho_1 [\cos \alpha_1 \cos \alpha + \cos \beta_1 \cos \beta + \cos \gamma_1 \cos \gamma] = p.$$

This becomes on applying (55),

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \tag{61}$$

which is the *normal equation.*

Exercise. — Show that for any point $P(x, y, z)$ not in the plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma \gtrless p.$$

175. Plane Parallel to One or More Coördinate Axes. —

Suppose the plane is given parallel to the z-axis. In this case the normal axis will lie in the xy-plane and $\gamma = \frac{\pi}{2}\cdot$ Hence (61) reduces to the form

$$x \cos \alpha + y \cos \beta = p. \tag{61a}$$

Similar equations are obtained for planes parallel to either of the other axes.

In case the plane is parallel to both the y- and z-axes, we have the normal along the x-axis. Then $\beta = \gamma = \frac{\pi}{2}$, and $\alpha = 0$. For this case (61) reduces to the form

$$x = p,$$

a result obvious from the definition of coördinates.

176. The General Equation of the First Degree. —

The normal equation (61) of the plane is of the first degree. We now wish to prove conversely that the locus of the general first-degree equation

$$Ax + By + Cz + D = 0 \tag{62}$$

is a plane. To do this we simply show that (62) can always be reduced to the form (61).

Dividing both sides of the equation by $\pm\sqrt{A^2+B^2+C^2}$, we have

$$\frac{A}{\pm\sqrt{A^2+B^2+C^2}}x+\frac{B}{\pm\sqrt{A^2+B^2+C^2}}y+\frac{C}{\pm\sqrt{A^2+B^2+C^2}}z=\frac{-D}{\pm\sqrt{A^2+B^2+C^2}}.$$

By § 171 the coefficients of x, y, and z are direction cosines of a line. Hence this equation has the same form as (61) and its locus is a plane. The direction cosines of its normal are

$$\frac{A}{\pm\sqrt{A^2+B^2+C^2}},\quad\frac{B}{\pm\sqrt{A^2+B^2+C^2}},\quad\frac{C}{\pm\sqrt{A^2+B^2+C^2}}$$

and the length of its normal intercept is

$$\frac{-D}{\pm\sqrt{A^2+B^2+C^2}}.$$

The sign of the radical is taken opposite to that of D so that the normal distance p shall be positive. Comparison with (61a) shows that if any one of the three variables is missing the locus is a plane parallel to the corresponding axis.

177. Angle between Two Planes. — The angle between two planes is readily seen to be the same as that between their normal axes. Hence the angle between two planes can be found by Formula 59. If we wish to express $\cos\theta$ in terms of the coefficients of (62), substitution of the above values of the direction cosines gives

$$\cos\theta=\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{A'^2+B'^2+C'^2}}. \qquad (63)$$

Parallel Planes. — If two planes are parallel, their normal axes are the same. Hence their direction cosines are the same or numerically equal with unlike signs. This gives

$$\frac{A}{\sqrt{A^2+B^2+C^2}}=\frac{\pm A'}{\sqrt{A'^2+B'^2+C'^2}},$$

$$\frac{B}{\sqrt{A^2+B^2+C^2}}=\text{etc.}$$

By alternation we have

$$\frac{A}{A'}=\frac{\sqrt{A^2+B^2+C^2}}{\sqrt{A'^2+B'^2+C'^2}},\qquad \frac{B}{B'}=\frac{\sqrt{A^2+B^2+C^2}}{\sqrt{A'^2+B'^2+C'^2}},$$

$$\frac{C}{C'}=\frac{\sqrt{A^2+B^2+C^2}}{\sqrt{A'^2+B'^2+C'^2}}.$$

This gives as the condition for parallelism,

$$\boldsymbol{\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}}. \tag{64}$$

Perpendicular Planes. — If the planes are perpendicular, $\cos\theta = 0$. Then from (63) we have at once

$$\boldsymbol{AA'+BB'+CC'=0}. \tag{65}$$

178. The Intercept Equation. — The *intercepts* of a plane are the distances from the origin to the points in which it meets the x-, y-, and z-axes. They are denoted by a, b, and c, respectively.

If in equation (62) we set $y = z = 0$, we find that the x-intercept, a, is $-\frac{D}{A}$. Similarly $b = -\frac{D}{B}$, and $c = -\frac{D}{C}$. Hence by transposing D and dividing both sides by $-D$, equation (62) may be reduced at once to the form

$$\boldsymbol{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1}, \tag{66}$$

which is known as the *intercept form* of the equation.

PROBLEMS

1. Write the equations of the following planes in the intercept and normal forms and determine which octant contains the foot of the normal axis:

(*a*) $3x - 4y + 12z - 5 = 0$;
(*b*) $4x + y + z - 8 = 0$;
(*c*) $4x - y - 8z + 10 = 0$;
(*d*) $15x - 8z + 20 = 0$;
(*e*) $2x + 9y - 6z = 18$;
(*f*) $4x + 7y - 4z + 27 = 0$.

2. The same as Problem 1 for the planes:

(*a*) $4x + 3y - 20 = 0$;
(*b*) $x + z - 1 = 0$;
(*c*) $x + y + z = 12$;
(*d*) $2x - 2y - z = 4$;
(*e*) $8x - 3y - 5z + 60 = 0$;
(*f*) $6x - 2y + 3z + 14 = 0$.

3. Write the equations of the planes determined by the following data:

(*a*) $a = 3, b = 4, c = -\frac{1}{2}$;

(*b*) $p = 5, \cos\alpha = \frac{1}{2}, \cos\beta = -\frac{2}{3}$;

(*c*) through the points $(1, 1, 1), (2, -1, -3), (4, 3, -2)$;

(*d*) parallel to the plane $3x - 4y + 8z - 5 = 0$ and containing the point $(2, 1, -4)$.

4. Write the equation of the plane:

(*a*) perpendicular to the plane $3x - 4y + 8z - 5 = 0$ and containing the points $(2, 1, -4)$ and the origin;

(*b*) which has the foot of the normal axis at the point $(-2, -4, 1)$;

(*c*) parallel to the z-axis and containing the points $(2, -4, 1)$ and $(3, 5, -2)$.

5. Find the angle between each of the planes of Problem 1 and the plane $x + 2y + 2z - 5 = 0$.

6. Find the angle between the planes of Problem 2 and the plane $x + 2y - 2z - 5 = 0$.

7. Find the angles at which each of the planes of Problem 2 meets the coördinate planes.

8. Show that the distance from the plane $x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$ to the point (x_1, y_1, z_1) is $x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p$.

9. Using the result of Problem 8 find the distance from each of the planes of Problem 2 to the point $(3, 0, -2)$.

10. Find the locus of points equidistant from the points $(-2, -3, 1)$ and $(4, 5, -7)$.

11. Find the point of intersection of the planes

$$\begin{array}{llll} (a) & x + 2y - 3z = 16, & (b) & 2x - 3y + 2z = 5, \\ & 3x - 4y - z = -7, & & 8x + 5y - 3z = 6, \\ & 2x - 8y + 5z = -41; & & 4x - 2y + z = 9. \end{array}$$

12. Show that the planes bisecting and perpendicular to the edges of the tetrahedron whose vertices are $(3, 1, -2)$, $(4, 2, -1)$, $(6, 1, -5)$, $(1, 3, 2)$ meet in a point.

13. Find the locus of points equidistant from the planes $2x - 3y + z = 4$ and $3x - 2y + z = 5$.

14. Find the volume of the tetrahedron whose vertices are given in Problem 12.

15. What is the locus of each of the following equations:

(*a*) $x^2 - 3x + 2 = 0$;
(*b*) $x^2 - y^2 = 0$;
(*c*) $x^2 + 2xy + y^2 - z^2 = 0$?

16. Name all pairs of the following planes which are (i) parallel; (ii) perpendicular to each other:

$$\begin{array}{llll} (a) & 2x - 3y + 5z = 8; & (d) & 3x - y - 3z = 10; \\ (b) & 4x + 6y + 2z = 1; & (e) & 7x - 2y - 4z = 14; \\ (c) & 6x - 9y + 15z = 5; & (f) & 6z + 2y - 6x = 15. \end{array}$$

179. The Equations of the Line. — Since any straight line may be regarded as the intersection of two planes, it will be seen from §§ 173 and 176 that it requires *two* equations of the form

$$Ax + By + Cz + D = 0$$

to determine a straight line.

It is more convenient, however, to determine a line from two given points upon it, or from one point and its direction angles. From these we derive equations for the line. It should be noted that in each case *two* equations are required.

180. The Symmetrical Equations. — Let the line pass through the point $P_1(x_1, y_1, z_1)$ and have direction angles α, β, and γ. If $P(x, y, z)$ is any point on the line and d denotes the distance P_1P, then by Formula 58

$$\cos\alpha = \frac{x - x_1}{d}, \qquad \cos\beta = \frac{y - y_1}{d}, \qquad \cos\gamma = \frac{z - z_1}{d}.$$

These relations give,

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}, \tag{67}$$

which are the *symmetrical equations.*

If a point P_1 and the direction numbers of the line a, b, and c are known, a more convenient form to use is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}. \tag{67a}$$

181. The Two-point Equations. — Let the line be determined by the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Then from (58) the difference of the respective coördinates are direction numbers. Hence by substituting $x_1 - x_2$, $y_1 - y_2$, and $z_1 - z_2$, for a, b, and c, respectively in (67a) we obtain the *two-point equations*

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}. \tag{67b}$$

182. The Projection Forms. — Let the equations of the projections of the given line on the xy- and xz-planes respectively be

$$\begin{aligned} y &= kx + m, \\ z &= lx + n. \end{aligned} \tag{68}$$

These equations taken together determine the line.

For each equation may be regarded as the equation of the projecting plane. For example, the first by (61a) is the equation of a plane parallel to the z-axis and hence perpendicular to the xy-plane. But it contains all points satisfying the relation $y = kx + m$ and so must contain the given projection. Similarly, the other equation is that of the plane projecting the line on the xz-plane. Thus the line is determined as the intersection of its projecting planes.

183. Reduction of the General Equations of a Line to the Standard Forms. — Consider the line determined by the equations

$$\begin{aligned} 4x + y - z + 2 &= 0, \\ x + 4y + 2z - 1 &= 0. \end{aligned}$$

To reduce these to the projection forms, eliminate first z and then y between the equations. We have at once,

$$3x + 2y + 1 = 0, \qquad 5x - 2z + 3 = 0.$$

That the locus of these equations is the same as the locus of those given is obvious, since coördinates satisfying the first pair of equations must satisfy the last.

To reduce the given equations to the symmetrical form solve the projection forms for x. We have,

$$x = \frac{2y + 1}{-3}, \qquad x = \frac{2z - 3}{5}.$$

Equating these, we have

$$\frac{x}{1} = \frac{y + \frac{1}{2}}{-\frac{3}{2}} = \frac{z - \frac{3}{2}}{\frac{5}{2}}.$$

The denominators are direction numbers, and if we divide each one by $\sqrt{1^2 + (-\frac{3}{2})^2 + (\frac{5}{2})^2} = \frac{1}{2} \cdot \sqrt{38}$ they become direction cosines by § 171. Doing so, we get

$$\frac{x}{\frac{2}{\sqrt{38}}} = \frac{y + \frac{1}{2}}{\frac{-3}{\sqrt{38}}} = \frac{z - \frac{3}{2}}{\frac{5}{\sqrt{38}}},$$

which are the symmetrical equations of a line having direction cosines $\frac{2}{\sqrt{38}}, \frac{-3}{\sqrt{38}}, \frac{5}{\sqrt{38}}$, and passing through the point $(0, -\frac{1}{2}, \frac{3}{2})$. But these are merely transformations of the original equations, and hence are the forms required.

PROBLEMS

1. Find the points at which the following lines cut the coordinate planes and draw the lines:

(*a*) $2x - 3y + 2z - 5 = 0, 3x - y - z + 6 = 0;$

(*b*) $2x - 4y = 5, 3y - z = 2;$

(*c*) $\frac{x-2}{6} = \frac{y-3}{2} = \frac{z+1}{3}.$

2. Do the lines of Problem 1 (*a*) and (*b*) meet?

3. Reduce the equations in Problem 1 to the symmetrical form.

4. Reduce the equations in Problem 1 to the projection form, the projecting planes being perpendicular to the *xz*- and *yz*-planes.

5. Find the equations of the straight lines determined as follows:

(*a*) through the points $(1, 0, -5)$ and $(-2, 3, 1)$;

(*b*) through the point (1, −1, 2) and parallel to the z-axis;

(*c*) through the point (1, −1, 2) and perpendicular to the z-axis;

(*d*) through the point (1, −1, 2) and having $\cos\alpha = \frac{1}{2}$, and $\cos\beta = \frac{1}{4}$;

(*e*) through the point (1, −1, 2) and parallel to the line of Problem 1 (*a*).

6. Show that if a line is perpendicular to a plane, it has the same direction cosines as the normal to the plane.

7. Find the angle between the lines

$$\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-5}{6} \quad \text{and} \quad \frac{x+3}{6} = \frac{y-2}{9} = \frac{z-4}{2}.$$

8. Show that if a line is parallel to a plane its direction cosines and those of the normal to the plane satisfy the relation

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0.$$

9. Find in the symmetrical form the equations of the locus of points

(*a*) equidistant from the points (3, −1, 2), (4, −6, −5), and (0, 0, −3);

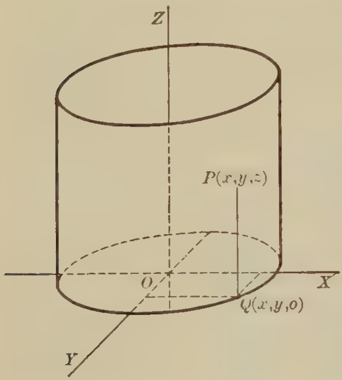
(*b*) equidistant from the planes $2x - 3y = 6$, $3x - 2z = 8$, and $2x + 3z = 5$.

10. Show that the line $\dfrac{x-3}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{1}$

(*a*) is parallel to the plane $6x - 8y + 12z - 60 = 0$;
(*b*) lies in the plane $3x - 4y + 6z - 19 = 0$;
(*c*) is perpendicular to the plane $2x + 3y + z = 6$.

184. Cylindrical Surfaces. — *A cylindrical surface is a surface generated by a moving straight line which constantly intersects a given fixed curve and remains parallel to a fixed straight line.* The fixed curve is called the *directrix* and the generating line the *generatrix.*

Let the directrix of a cylindrical surface be an ellipse in



the xy-plane symmetrical with respect to the coördinate axes, and let the generatrix remain parallel to the z-axis. Then the equation of the ellipse in its plane will be

$$b^2x^2 + a^2y^2 = a^2b^2.$$

Let $P(x, y, z)$ be any point on the surface, and Q the corresponding point on the ellipse. Evidently the coördinates of Q will be $(x, y, 0)$; *i.e.*, the x- and y-coördinates of every point on the line PQ are the same as those of Q. But the coördinates of Q satisfy the equation of the ellipse. Hence for any point on the cylindrical surface,

$$b^2x^2 + a^2y^2 = a^2b^2.$$

The preceding is perfectly general, and we see that it leads at once to the theorem:

A cylindrical surface whose directrix is a curve in the xy-plane and whose generatrix moves parallel to the z-axis, has the same equation as the directrix.

The converse, which is readily established, is:

The locus of an equation in two variables, x and y, is a cylindrical surface, whose directrix is the curve in the xy-plane which has the same equation, and whose generatrix moves parallel to the z-axis.

Similar theorems hold of course for cylinders whose generatrices are parallel to either the x- or y-axes.

Exercise. What is the locus of any equation in *one* variable?

185. Surfaces of Revolution. — *A surface generated by revolving a plane curve about a fixed line in its plane as an axis is called a surface of revolution.*

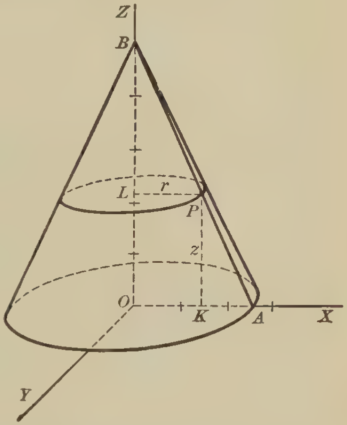
The curve is called the *generatrix;* its position with reference to the axis is unchanged during the revolution. Sections of the surface made by planes through the axis are called *meridional sections.* From the definition of such a surface it is evident that

(a) sections made by planes perpendicular to the axis are circles;

(b) any meridional section is the generatrix itself.

Let us first consider a conical surface generated by revolving a straight line about the z-axis. Let the position of the generatrix in the xz-plane be AB, of which the equation is

$$2x + z = 5.$$



Let $P(x, y, z)$ be any point of the locus. As the line turns about the z-axis, P describes a circle of radius $r = LP$ in a plane parallel to the xy-plane and distant from it $KP = z$. Now when the line is in the xz-plane, $x = r$; hence for all positions of the line,

$$2r + z = 5.$$

But, as P describes a circle of radius r in a plane parallel to the xy-plane, we have at once

$$x^2 + y^2 = r^2.$$

Substituting above, we have

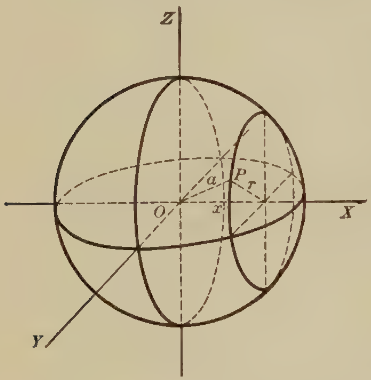
$$2\sqrt{x^2 + y^2} + z = 5.$$

Simplifying, we have the equation

$$4(x^2 + y^2) = (z - 5)^2.$$

Consider the surface generated by revolving about the x-axis a circle of radius a whose center is the origin. The equation of the generatrix in the xz-plane is

$$x^2 + z^2 = a^2.$$



Evidently the point P describes a circle with its center on the x-axis and of radius r; hence

$$x^2 + r^2 = a^2.$$

But $y^2 + z^2 = r^2$.

Hence we have at once,

$$\mathbf{x^2 + y^2 + z^2 = a^2.} \quad \textbf{(69)}$$

This equation is important, as it is the equation of a sphere of radius a and having its center at the origin.

PROBLEMS

1. Find the equation of the surface generated by revolving the curve about the axis indicated:

(*a*) $3x + 12y = 3$, x-axis;
(*b*) $z^2 = 2px$, x-axis;
(*c*) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, y-axis.

2. Identify and sketch the following surfaces:

(*a*) $x^2 - 4y^2 = 4$;
(*b*) $x^2 + 2z^2 = 16$;
(*c*) $y^2 + z^2 + 2y + 6z = 15$;
(*d*) $xz = 15$;
(*e*) $y^2 + 4y - 5 = 0$;
(*f*) $x^2 = 16z$.

3. Find the equation of the surface of a cone whose vertex is at the origin, with the z-axis for its axis of revolution and opposite elements perpendicular to each other.

4. Show that the locus of a point equidistant from the point $(p, 0, 0)$ and the yz-plane is the paraboloid of revolution $y^2 + z^2 = 2\,px - p^2$.

5. Find the equation of the locus of a point equidistant from a given plane and a given line parallel to the plane.

Hint. — Let $x = 0$ be the given plane and $x = p$, $y = 0$ be the given line.

6. Show that the locus of a point in space the sum of whose distances from the points $(\pm c, 0, 0)$ is $2\,a$ is the prolate spheroid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$.

7. Find the equation of the surface generated:

(a) by revolving about the z-axis the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$;

(b) by revolving about the x-axis the ellipse $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$.

8. How may the hyperboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1$ be defined as a locus?

9. Find the equation of the locus of points in space such that the difference of its distances from the points $(0, 0, \pm c)$ is $2\,a$.

10. Find the equation of the surface generated:

(a) by revolving about the x-axis the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$;

(b) by revolving about the z-axis the ellipse $\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$;

(c) by revolving about the y-axis the parabola $x^2 + 2\,py = 0$.

186. Discussion of Surfaces. — The discussion of a surface is in general a much more complicated matter than that of a curve. The notions of intercepts, symmetry, and extent are readily generalized. Besides these, however, we have two other methods which help to determine the nature of a surface. The first is to find the section of the surface by any of the coördinate planes; the second is to find the section of the surface by any plane parallel to one of the coördinate planes. The former is called a *trace* of the surface.

To illustrate these, consider the sphere, with equation

$$x^2 + y^2 + z^2 = a^2.$$

To find the equations of the traces, set x, y, and z successively equal to 0. When $x = 0$, we have

$$y^2 + z^2 = a^2.$$

Thus the yz-trace is a circle of radius a. The xy- and xz-traces are also circles of the same radius, with equations

$$x^2 + y^2 = a^2$$

and $$x^2 + z^2 = a^2 \text{ respectively.}$$

The section made by a plane parallel to the yz-plane and at a distance c from it is the curve determined by the given equation and the equation $x = c$. Eliminating x between the two equations, we have

$$y^2 + z^2 = a^2 - c^2,$$

which shows that the section is a circle having the point $(c, 0, 0)$ as a center and $\sqrt{a^2 - c^2}$ as the radius. If $c > a$, this is imaginary, indicating that the surface lies wholly within the planes $x = \pm a$. A similar discussion holds for sections parallel to the other coördinate planes.

To sketch a surface when only two variables of its equation are of the same degree, or of the same degree and sign, first draw sections parallel to the plane of the two variables. These sections and the traces give the best representation of the locus.

187. Quadric Surfaces. — By analogy to the conics, the locus of the general equation of the second degree

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Kz + L = 0$$

is called a *quadric surface.* It can be shown that these surfaces have properties analogous to those of the conics; and in particular that every plane section of such a surface is a conic. We shall not go into these details, but will confine our discussion to the simpler forms to which the general equation may be reduced.

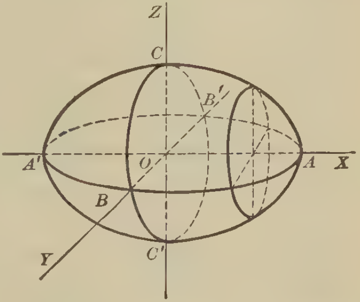
188. The Ellipsoid. — The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (70)$$

is called an ellipsoid.

From the form of the equation we see at once that the surface is symmetrical with respect to all of the coördinate planes and coördinate axes. The intercepts on the axes are

$$x = \pm a,$$
$$y = \pm b,$$
$$z = \pm c.$$



Setting $x = 0$, the equation of the trace on the yz-plane is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

hence it is an ellipse with semi-axes b and c. In like manner the traces on the other coördinate planes are ellipses.

Setting $x = k$, the equation becomes $\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2 - k^2}{a^2}$, a form which shows that there is no section for k numerically greater than a. Dividing by the right-hand member, this becomes

$$\frac{y^2}{\frac{b^2}{a^2}(a^2 - k^2)} + \frac{z^2}{\frac{c^2}{a^2}(a^2 - k^2)} = 1.$$

Hence the various sections are ellipses symmetrical with respect to the x-axis. The semi-axes, $\frac{b}{a}\sqrt{a^2 - k^2}$ and $\frac{c}{a}\sqrt{a^2 - k^2}$, grow smaller as k increases, until the ellipse becomes a point when $k = a$. Similarly the sections parallel to the other coördinate planes are ellipses.

The values a, b, and c are called the semi-axes of the ellipsoid. When $a = b = c$, the ellipsoid is evidently a sphere with the center at the origin. Ordinarily the semi-axes are unequal, and in this form of the equation it is usually assumed that $a > b > c$. When $b = c$, but $a > b$ or c, the ellipsoid is called a prolate spheroid. In this case the sections parallel to the yz-plane are circles. The locus is then a surface of revolution since it is generated by revolving the xz-trace (or the xy-trace) about the x-axis. When $a = b$, but $c < a$ or b, the surface is that of an oblate spheroid. This is also a surface of revolution and the sections parallel to the xy-plane are circles.

189. Hyperboloid of One Sheet. — The locus of the equation

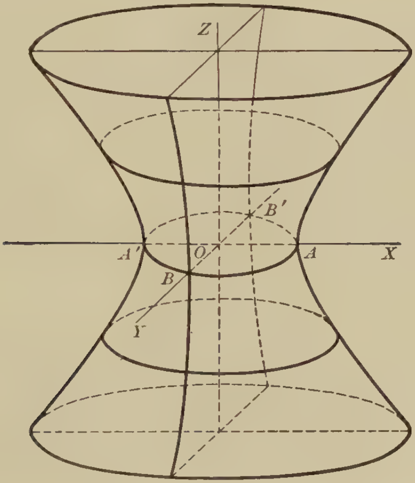
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (71)$$

is called a hyperboloid of one sheet.

In this case sections parallel to the xy-plane, made by the planes $z = k$, have equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}$$

and hence are ellipses which increase in size as the numerical value of k increases.



The xz-trace and the yz-trace are the hyperbolas

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad \text{and} \qquad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ respectively.}$$

If $a = b$, the locus is a surface of revolution about the z-axis.

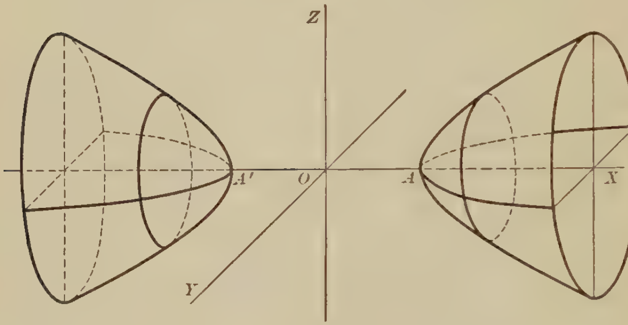
190. Hyperboloid of Two Sheets. — The locus of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (72)$$

is called the hyperboloid of two sheets.

In this case we consider sections parallel to the yz-plane made by the planes $x = k$. These are the ellipses

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1.$$



If $k^2 < a^2$ the right-hand member is negative. Hence no part of the locus is between the two planes $x = \pm a$. If $k = \pm a$, each corresponding section is a point ellipse. As k increases numerically from a to ∞ the section increases indefinitely.

The xy-trace and the xz-trace are the hyperbolas

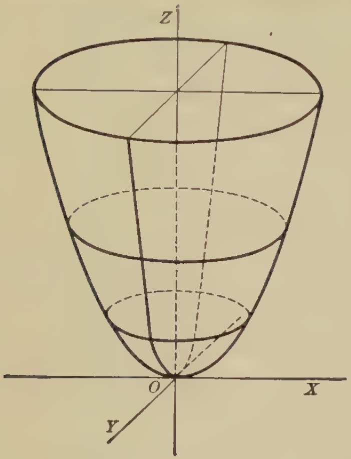
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \text{ respectively.}$$

If $b = c$, the locus is a surface of revolution.

191. Elliptic Paraboloid. — This is the locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2\,cz. \qquad (73)$$

Proceeding as before, we observe that sections made by the planes $z = k$ are ellipses whose axes increase indefinitely as k increases. The surface lies wholly above or wholly below the xy-plane, according as c is positive or negative. The xz- and yz-traces are parabolas. If $a = b$, the locus is a surface of revolution.

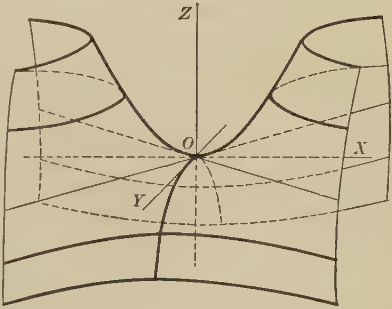


192. Hyperbolic Paraboloid. — This is the locus of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,cz. \quad (74)$$

Consider c positive. Sections made by the planes $z = k$ are the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,ck.$$



As k increases from 0 to ∞ the vertices of the corresponding sections lie in the xz-plane and recede indefinitely from the z-axis. As k decreases from 0 to $-\infty$ the vertices of the sections are in the yz-plane and recede indefinitely from the z-axis. The xz- and yz-traces are parabolas, each having its vertex at the origin, the former extending above the xy-plane, the latter below. The xy-trace is a pair of lines intersecting at the origin.

PROBLEMS

1. Find the traces of the following surfaces on each of the coördinate planes:

(*a*) $x^2 + y^2 = 12z$; (*d*) $x^2 - 4y^2 - 4z^2 = 9$;
(*b*) $x^2 + 4y^2 + 4z^2 = 16$; (*e*) $x^2 = y^2 + 9z^2$;
(*c*) $x^2 - 4y^2 = 0$; (*f*) $4x^2 + 9y^2 - 9z^2 = 36$.

2. The same as Problem 1 for the surfaces:

(*a*) $x^2 + 4z^2 = 16y$; (*d*) $x^2 + y^2 = 9z^2$;
(*b*) $y^2 - 4z^2 = 0$; (*e*) $4x^2 - 9y^2 - 9z^2 = 36$;
(*c*) $x^2 + 4y^2 + 9z^2 = 25$; (*f*) $9x^2 + 4y^2 - 16z^2 = 64$.

3. Find the intersection of each of the surfaces in Problem 1 with the plane $x = 4$.

4. Find the intersection of each of the surfaces in Problem 2 with the plane $y = 3$.

5. Discuss the traces of the surfaces:

(*a*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; (*b*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$.

6. Discuss the traces of the surfaces:

(*a*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; (*b*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} + 2z = 0$.

7. Show that the sections of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ are parabolas if perpendicular to the x- or the y-axis, and ellipses if perpendicular to the z-axis and $z > 0$.

8. Show that the sections of the surface $\frac{x^2}{a^2} - \frac{y^2}{b^2} + 2z = 0$ perpendicular to the x- or the y-axis are parabolas. How do those perpendicular to the x-axis differ from those perpendicular to the y-axis?

9. Discuss and sketch the following surfaces:

(*a*) $4x^2 + 25y^2 + 16z^2 = 100$;
(*b*) $28x^2 + 196 = 9y^2 + 16z^2$;

(*c*) $4x^2 + 9y^2 - z^2 + 36 = 0$;
(*d*) $16x^2 + y^2 + 64z = 0$;
(*e*) $9z^2 + 36y^2 - 81x = 0$;
(*f*) $y^2 - z^2 + 8x = 0$.

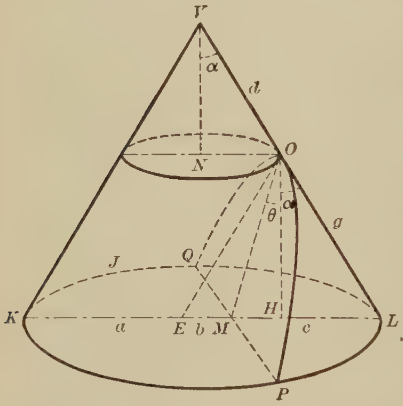
10. Discuss and sketch the following surfaces:

(*a*) $x^2 + 4y^2 + 9z^2 = 36$;
(*b*) $12y^2 + 144 = 16x^2 + 25z^2$;
(*c*) $9x^2 - 16y^2 + 4z^2 + 144 = 0$;
(*d*) $4x^2 - z^2 = 16y$;
(*e*) $4y^2 + 9z^2 = 18x$;
(*f*) $x^2 - 2y^2 + 8z = 0$.

APPENDIX

Theorem. — *Any plane section of a right circular cone is an ellipse, parabola, hyperbola, or a limiting form of one of these curves. Hence they are called conic sections, or conics.*[*]

Let V be the vertex of a right circular cone whose axis VN makes an angle α with the elements of the cone. Let JKL be any circular section perpendicular to the axis, and having KL for a diameter. Through O, any point on the element VL pass a plane POQ intersecting the plane JKL in the line PQ perpendicular to KL.



Draw OH and OE parallel to VN and VK respectively and draw OM. Let θ be the angle which the plane POQ makes with the vertical line OH, and let d be the distance of the vertex V from O. Call $OE\ (= OL) = g$, $KE = a$, $EM = b$, and $ML = c$. Also in the plane POQ, let O be the origin of coördinates and OM the x-axis, so that the coördinates of P are $OM = x$, $MP = y$.

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[*] The authors are indebted to Professor W. W. Landis for the following proof of this theorem.

By plane geometry, MP is a mean proportional between KM and ML, hence,

$$y^2 = (a \pm b)c \tag{1}$$

according as M is to the right or left of the point E.

Also since $MH = \frac{1}{2}(c \mp b)$, and $OM = x$, we have

$$x^2 = \left(\frac{c \mp b}{2}\right)^2 + \overline{OH}^2.$$

and
$$g^2 = \left(\frac{c \pm b}{2}\right)^2 + \overline{OH}^2.$$

Subtracting these $x^2 = g^2 \mp bc$.

Adding this to the above equation, we have

$$x^2 + y^2 = g^2 + ac. \tag{2}$$

Now $OH = g \cos \alpha = x \cos \theta$, therefore $g = \dfrac{x \cos \theta}{\cos \alpha}$.

Also $a = 2\,NO = 2\,d \sin \alpha$,

and
$$\begin{aligned} c &= MH + HL = x \sin \theta + g \sin \alpha \\ &= (\sin \theta + \cos \theta \tan \alpha)x; \end{aligned}$$

hence $ac = 2\,d \sin \alpha\,(\sin \theta + \cos \theta \tan \alpha)x$. Substituting these values for g and ac, equation (2) becomes

$$x^2 + y^2 = \left(\frac{\cos \theta}{\cos \alpha}\right)^2 x^2 + 2\,d \sin \alpha\,(\sin \theta + \cos \theta \tan \alpha)x,$$

or

$$\left(1 - \frac{\cos^2 \theta}{\cos^2 \alpha}\right)x^2 + y^2 - 2\,d \sin \alpha\,(\sin \theta + \cos \theta \tan \alpha)x = 0. \tag{3}$$

Using the formulas for $\sin 2\,\alpha$ and $\sin\,(\alpha + \theta)$ this equation reduces readily to

$$(\cos^2 \alpha - \cos^2 \theta)x^2 + \cos^2 \alpha \cdot y^2 - d \sin 2\,\alpha \sin\,(\alpha + \theta) \cdot x = 0. \tag{4}$$

For different values of the parameter θ we distinguish the following cases:

(*a*) If $\theta = 90°$ equation (4) becomes

$$x^2 + y^2 - 2\,d \sin \alpha \cdot x = 0$$

which is a circle with center at N since $ON = d \sin \alpha$.

(*b*) If $\theta > \alpha$, $\cos \theta < \cos \alpha$ and the coefficients of x^2 and y^2 have the same sign. The plane POQ cuts all the elements of the cone and the section is an ellipse.

(*c*) If $\theta = \alpha$ equation (4) becomes

$$y^2 - 4\,d \sin^2 \alpha \cdot x = 0.$$

The plane POQ is then parallel to the element VK and the section is a parabola with the distance of the focus from the vertex equal to $d \sin^2 \alpha$.

Exercise. — Show that the focus of this parabola is the foot of the perpendicular on the plane POQ drawn from N.

(*d*) If $\theta < \alpha$ the coefficients of x^2 and y^2 have unlike signs and the equation is that of the hyperbola. If the elements of the given cone be produced through the vertex to form an equal inverted cone, the intersection of these two cones by the plane POQ gives the two branches of the hyperbola.

Limiting Forms. — The special cases or limiting forms of the conics may be obtained from the equations above by giving special values to the parameter d. These will be considered in order.

(*a*) As d approaches zero the equation of the circle approaches the limiting form $x^2 + y^2 = 0$. The plane POQ passes through the vertex and the section becomes a point-circle. If the plane is moved farther, the locus is on the inverted cone.

(*b*) This is the same case as (*a*) excepting the plane section through the vertex is known as a point-ellipse.

The coefficients of x^2 and y^2 differ but have the same sign.

(*c*) When $d = 0$, the parabolic section becomes $y^2 = 0$. The plane *POQ* becomes a tangent plane cutting the cone in a double straight line. If it is moved still farther the locus is on the inverted cone.

(*d*) The limiting form of the hyperbolic section as d approaches zero is two intersecting straight lines. The coefficients of x^2 and y^2 differ in sign, and since these are the only terms remaining in the equation, it has two linear factors. The plane *POQ* passing through the vertex intersects in two elements of the cone.

(*e*) The only remaining case to be considered is when the vertex of the cone is moved away indefinitely so that the cone approaches a cylinder as a limiting form. The parabolic section is then a pair of parallel lines.

FORMULAS AND EQUATIONS

Formulas of Distance and Direction

		No.	Page
Distance:	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$	(1)	8
Point dividing line in the ratio $r_1 : r_2$:	$x_0 = \frac{r_2x_1 + r_1x_2}{r_1 + r_2}, \quad y_0 = \frac{r_2y_1 + r_1y_2}{r_1 + r_2}.$	(2)	9
Mid-point:	$x_0 = \frac{1}{2}(x_1 + x_2), \quad y_0 = \frac{1}{2}(y_1 + y_2).$	(2a)	9
Slope:	$m = \frac{y_1 - y_2}{x_1 - x_2}.$	(3)	15
Test for parallelism:	$m_1 = m_2.$	(4)	18
Test for perpendicularity:	$m_1m_2 = -1.$	(5)	18
Angle between two lines:	$\tan \beta = \frac{m_1 - m_2}{1 + m_1m_2}.$	(6)	18

The Straight Line

Equations:			
Point slope form:	$y - y_1 = m(x - x_1).$	(7)	51
Two point form:	$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$	(7a)	52
Slope intercept form:	$y = mx + b.$	(8)	52
Intercept form:	$\frac{x}{a} + \frac{y}{b} = 1.$	(9)	53
General form:	$Ax + By + C = 0.$	(10)	56
Test for parallelism:	$A : A' = B : B'.$		57

	No.	Page
Test for perpendicularity: $AA' = -BB'$.		58
Test for identity: $A : A' = B : B' = C : C'$.		59
Normal form: $x \cos \omega + y \sin \omega - p = 0$.	(11)	65
To reduce form (10) to form (11), divide by $\pm\sqrt{A^2 + B^2}$, sign to agree with that of B.		66
Distance from a line to a point: $d = x_1 \cos \omega + y_1 \sin \omega - p$.	(12)	69

The Circle

	No.	Page
Equations:		
Standard form: $(x - h)^2 + (y - k)^2 = r^2$.	(13)	82
Center at origin: $x^2 + y^2 = r^2$.	(13*a*)	82
General form: $x^2 + y^2 + Dx + Ey + F = 0$.	(14)	85
Length of a tangent from P_1: $t^2 = (x_1 - h)^2 + (y_1 - k)^2 - r^2$.	(15)	93
$t^2 = x_1^2 + y_1^2 + Dx_1 + Ey_1 + F$.	(15*a*)	94
Equation of the radical axis: $(D - D')x + (E - E')y + F - F' = 0$.	(16)	97

The Parabola

	No.	Page
Equations: $y^2 = 2\,px$.	(17)	102
$x^2 = 2\,py$.	(17*a*)	102
Latus rectum: $2\,p$.		103

The Ellipse

	No.	Page
Formulas connecting the fundamental constants: $a^2 = b^2 + c^2$.	(19)	113
$c = ae$.	(21)	117

		No.	Page
Equations:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$	(18)	112
	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$	(18*a*)	116
Focal radii:	$\rho + \rho' = 2\,a.$	(20)	113
Latus rectum:	$\frac{2\,b^2}{a}.$		113

The Hyperbola

Formulas connecting the fundamental constants:			
	$c^2 = a^2 + b^2.$	(23)	123
	$c = ae.$	(21)	127
Equations:	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$	(22)	123
	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$	(22*a*)	124
Latus rectum:	$\frac{2\,b^2}{a}.$		124
Focal radii:	$\rho' - \rho = 2\,a.$	(24)	124
Equations of the asymptotes:			
	$y = \pm\frac{b}{a}x.$	(25)	128
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$	(25*a*)	128
Equilateral hyperbolas:			
	$x^2 - y^2 = \pm a^2.$	(26)	130
	$xy = \pm\frac{a^2}{2}.$	(26*a*)	130

Transformation of Coördinates

Translation:	$x = x' + h,\ y = y' + k.$	(27)	136
Rotation:	$x = x' \cos\theta - y' \sin\theta,$ $y = x' \sin\theta + y' \cos\theta.$	(28)	137

	No.	Page
Equation of the conic with the directrix as the y-axis and the focus on the x-axis:		
$(1 - e^2)x^2 - 2\,px + y^2 + p^2 = 0.$	(29)	145
Generalized standard equations of the conics:		
Parabola: $(y - k)^2 = 2\,p(x - h).$	(30)	147
$(x - h)^2 = 2\,p(y - k).$	(30a)	147
Ellipse: $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1.$	(31)	147
$\dfrac{(y - k)^2}{a^2} + \dfrac{(x - h)^2}{b^2} = 1.$	(31a)	147
Hyperbola: $\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1.$	(32)	147
$\dfrac{(y - k)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1.$	(32a)	147
Angle of rotation for eliminating the xy term:		
$\tan 2\,\theta = \dfrac{B}{A - C}\cdot$	(33)	151

Polar Coördinates

	No.	Page
Relation to rectangular coördinates:		
$x = \rho \cos \theta,\ y = \rho \sin \theta.$	(34)	161
$\rho^2 = x^2 + y^2,\ \theta = \operatorname{arc} \tan \dfrac{y}{x}\cdot$	(35)	161
Equations of the straight line:		
$\rho \cos \theta = a.$	(36)	164
$\rho \sin \theta = a.$	(36a)	164
$\theta = c.$	(37)	165
Equations of the circle:		
$\rho = r.$	(38)	165
$\rho = 2\,r \cos \theta.$	(39)	165
$\rho = 2\,r \sin \theta.$	(39a)	165
$\rho = a \cos \theta + b \sin \theta.$	(40)	165

	No.	Page
Formula of rotation: $\theta = \theta' + \alpha$.	(41)	171
Equations of the conics:		
$\rho = \frac{ep}{1 - e \cos \theta}$.	(42)	172
$\rho = \frac{ep}{1 - e \sin \theta}$.	(42*a*)	172

Tangents and Normals

	No.	Page
Slope of a tangent:		
To the circle: $m = -\frac{x_1}{y_1}$.	(43)	211
To the ellipse: $m = -\frac{b^2x_1}{a^2y_1}$.	(44)	212
To the parabola: $m = \frac{p}{y_1}$.	(45)	212
To the hyperbola: $m = \frac{b^2x_1}{a^2y_1}$.	(46)	212
Length of the subtangent: $\frac{y_1}{m}$.	(47)	215
Length of the subnormal: my_1.	(48)	216
Equation of the diameter bisecting chords of slope m:		
In the ellipse: $y = -\frac{b^2}{a^2m}x$.	(49)	221
In the parabola: $y = \frac{p}{m}$.	(51)	222
In the hyperbola: $y = \frac{b^2}{a^2m}x$.	(52)	222
Relation between the slopes of conjugate diameters:		
In the ellipse: $mm' = -\frac{b^2}{a^2}$.	(50)	222
In the hyperbola: $mm' = \frac{b^2}{a^2}$.	(53)	222

SOLID ANALYTIC GEOMETRY

	No.	Page
Radius vector and direction cosines:		
$\rho = x^2 + y^2 + z^2.$	(54)	260
$x = \rho \cos \alpha, \quad y = \rho \cos \beta, \quad z = \rho \cos \gamma.$	(55)	261
$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$	(56)	261
Distance:		
$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$	(57)	261
Direction cosines:		
$\cos \alpha = \frac{x_1 - x_2}{d}, \ \cos \beta = \frac{y_1 - y_2}{d}, \ \cos \gamma = \frac{z_1 - z_2}{d}.$	(58)	262
Angle between two lines:		
$\cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2.$	(59)	263
Condition for parallelism:		
$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_2, \quad \gamma_1 = \gamma_2,$		
or $\alpha_1 = \pi - \alpha_2, \quad \beta_1 = \pi - \beta_2, \quad \gamma_1 = \pi - \gamma_2.$		
Condition for perpendicularity:		
$\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2 = 0.$	(60)	264
Equations of the plane:		
Normal form: $x \cos \alpha + y \cos \beta + z \cos \gamma = p.$	(61)	267
Parallel to the z-axis: $x \cos \alpha + y \cos \beta = p.$	(61*a*)	267
General form:		
$Ax + By + Cz + D = 0.$	(62)	267
Angle between two planes:		
$\cos \theta = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2} \cdot \sqrt{A'^2 + B'^2 + C'^2}}.$	(63)	268
Test for parallelism:		
$A : A' = B : B' = C : C'.$	(64)	269
Test for perpendicularity:		
$AA' + BB' + CC' = 0.$	(65)	269
Intercept equation of the plane:		
$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$	(66)	269

		No.	Page
Equations of the line:			
Symmetrical forms:	$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\cos\beta}=\frac{z-z_1}{\cos\gamma}$.	(67)	272
	$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$.	(67*a*)	272
Two point forms:	$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$.	(67*b*)	272
Projection forms:	$y=kx+m,\ z=lx+n$.	(68)	273
Equations of the quadric surfaces:			
Sphere:	$x^2+y^2+z^2=a^2$.	(69)	278
Ellipsoid:	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.	(70)	281
Hyperboloid of one sheet:	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.	(71)	282
Hyperboloid of two sheets:	$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.	(72)	283
Elliptic paraboloid:	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2cz$.	(73)	284
Hyperbolic paraboloid:	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2cz$.	(74)	285

ANSWERS

The following tables of answers cover the majority of the odd-numbered problems. The exceptions consist mainly of cases where a statement of the answer would destroy the usefulness of the problem or where it is very easy to check the solution. Radicals and fractions which involve large numbers have been evaluated to two places of decimals; angles are given correct to minutes.

Pages 4, 5, 6

5. (*a*) 36; (*b*) 26; (*c*) 48; (*d*) $\frac{1}{2}(a-d)(b-c)$.

7. (*a*) 12.04; (*b*) 13.60; (*c*) 14.42; (*d*) $\sqrt{(a-d)^2+(b-c)^2}$.

17. $\left(\pm\frac{a\sqrt{2}}{2}, 0\right), \left(0, \pm\frac{a\sqrt{2}}{2}\right)$.

19. $\left(\frac{a\sqrt{3}}{2}, \pm\frac{a}{2}\right)$ or $\left(-\frac{a\sqrt{3}}{2}, \pm\frac{a}{2}\right)$.

21. (*a*) $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right), \left(\frac{3a}{2}, \frac{a\sqrt{3}}{2}\right)$; (*b*) $\left(\pm\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$.

23. $(a+b, c)$ or $(b-a, c)$ or $(a-b, -c)$.

Pages 12, 13

3. (*a*) 15.35; (*b*) 30.20; (*c*) 29.05; (*d*) 29.05; (*e*) 75.80; (*f*) $2a+2\sqrt{a^2+b^2}$.

5. $3\sqrt{6}$. **7.** 84.5. **9.** $17\sqrt{2}$.

11. (*a*) $(11, -\frac{9}{2})$; (*b*) $(2, -\frac{7}{4})$; (*c*) $(\frac{3}{4}, \frac{1}{2})$; (*d*) $(\frac{3}{4}a, \frac{3}{4}b)$.

13. (*a*) $(1, \frac{10}{3})$; (*b*) $(-\frac{2}{3}, \frac{4}{3})$; (*c*) $(-\frac{1}{3}, 0)$; (*d*) $(0, -\frac{1}{3})$; (*e*) $\left(\frac{13}{3}, -\frac{16}{3}\right)$; (*f*) $\left(0, \frac{b}{3}\right)$.

15. (1, 3).

17. $(2+3\sqrt{3}, 1+\sqrt{3})$ or $(2-3\sqrt{3}, 1-\sqrt{3})$.

Pages 19, 20, 21

1. (*a*) $\frac{3}{2}$, $56^\circ 19'$; (*b*) $\frac{14}{9}$, $57^\circ 16'$; (*c*) $-\frac{41}{34}$, $129^\circ 40'$;
(*d*) $\frac{15}{13}$, $49^\circ 5'$; (*e*) $5 - 2\sqrt{6}$, $5^\circ 46'$; (*f*) -1, 135°.

7. (*a*) $6^\circ 20'$; (*b*) $83^\circ 40'$; (*c*) $8^\circ 40'$;
(*d*) $8^\circ 40'$; (*e*) $69^\circ 38'$; (*f*) $19^\circ 33'$.

9. (*a*) 90°; (*b*) 0°; (*c*) 75°; (*d*) 75°; (*e*) $14^\circ 2'$; (*f*) $64^\circ 6'$.

11. $86^\circ 11'$.

13. (*a*) $26^\circ 34'$, $153^\circ 26'$; (*b*) $26^\circ 34'$, $153^\circ 26'$;
(*c*) $2^\circ 56'$, $177^\circ 4'$; (*d*) $52^\circ 15'$; $127^\circ 45'$.

15. (*a*) 45°, 45°, 90°; (*b*) $78^\circ 41'$, $78^\circ 41'$, $22^\circ 37'$;
(*c*) $50^\circ 51'$, $61^\circ 52'$, $67^\circ 17'$; (*d*) $19^\circ 26'$, $21^\circ 10'$, $139^\circ 24'$.

21. (0.211, 3.894) or (3.789, 2.106).

Pages 31, 32, 33

3. (*a*) $6x - 8y = 27$; (*b*) $20x - 6y = 11$;
(*c*) $10x + 8y + 9 = 0$; (*d*) $x - y + 6 = 0$;
(*e*) $3x - y = 10$; (*f*) $2x = a$.

5. (*a*) $x^2 + y^2 + 10x - 12y + 52 = 0$;
(*b*) $x^2 + y^2 - 2x + 6y - 26 = 0$;
(*c*) $x^2 + y^2 = 100$;
(*d*) $(x - h)^2 + (y - k)^2 = r^2$.

7. (*a*) $x^2 - 6x + 12y = 27$;
(*b*) $y^2 + 12x - 12 = 0$;
(*c*) $y^2 + 10y - 6x + 16 = 0$;
(*d*) $y^2 = 2px - p^2$.

9. (*a*) $x^2 - 3y^2 - 6x + 48y = 135$;
(*b*) $3x^2 - y^2 - 36x + 60 = 0$;
(*c*) $3x^2 - y^2 + 24x - 10y + 11 = 0$;
(*d*) $3x^2 - y^2 + 2px - p^2 = 0$.

11. (*a*) $4x^2 + 3y^2 - 24x + 12y = 0$;
(*b*) $3x^2 + 4y^2 + 24x = 0$;
(*c*) $3x^2 + 4y^2 - 6x + 40y + 91 = 0$;
(*d*) $3x^2 + 4y^2 - 8px + 4p^2 = 0$.

13. (*a*) $y^2 + 16x = 32$;
(*b*) $y^2 - 10x - 4y - 1 = 0$;
(*c*) $x^2 - 16y + 16 = 0$;
(*d*) $x^2 + 2(p + 2)y = (p + 2)^2$.

15. (*a*) $3x + y = 13$; (*b*) $3x + 5y = 27$;
(*c*) $x = y$; (*d*) $bx + ay = ab$.

17. (*a*) $x - y + 5 = 0$; (*b*) $x + y = 1$;
(*c*) $x + y\sqrt{3} = 3\sqrt{3} - 2$; (*d*) $x - y\sqrt{3} = -3\sqrt{3} - 2$.

19. $x^2 + y^2 = c^2$. **21.** $2xy = k$.

Page 48

5. (*b*) $k > 20$, $k = 20$, $k < 20$;
(*c*) $k < -13$, $k = -13$, $k > -13$.

Pages 49, 50

3. $(-1.69, -4.77)$. **5.** $(3.83, 4.17)$, $(-7.83, 15.83)$.

7. $(\pm 3, -4)$, $(\pm 4.90, 1)$. **9.** $(\pm 1.97, -0.5)$.

11. $(4, \pm 7.21)$, $(1, \pm 6.08)$.

13. $(2.83, 1.41)$, $(-2.83, -1.41)$, $(3.46, 1.15)$, $(-3.46, -1.15)$.

15. $(0, 0)$, $(2, \pm 2.83)$.

17. $(1.41, 2.83)$, $(-1.41, -2.83)$, $(2.83, 1.41)$, $(-2.83, -1.41)$.

19. $(\pm 2a, a)$. **21.** $(1.46, 2.91)$, $(-1.46, -2.91)$.

23. $\left(\dfrac{r}{\sqrt{1+m^2}}, \dfrac{mr}{\sqrt{1+m^2}}\right), \left(\dfrac{-r}{\sqrt{1+m^2}}, \dfrac{-mr}{\sqrt{1+m^2}}\right)$.

25. $(4.44, -3.89)$. **27.** $(5, 2)$, $(-5, -2)$.

29. $\left(\dfrac{4}{\sqrt{1-m^2}}, \dfrac{4m}{\sqrt{1-m^2}}\right), \left(\dfrac{-4}{\sqrt{1-m^2}}, \dfrac{-4m}{\sqrt{1-m^2}}\right)$.

31. -1, 7.07.

Pages 54, 55, 56

3. (*a*) $3x - 8y + 22 = 0$; (*b*) $7x + 2y = 32$;
(*c*) $8x - 11y = 1$; (*d*) $2x + 3y = 1$;
(*e*) $8x - y = 70$; (*f*) $13x + 10y = 34$;
(*g*) $2x - 7y = 12$; (*h*) $7x + 3y = 34$.

5. (*a*) $x - y = 2$; (*b*) $x + y = 3$;
(*c*) $x\sqrt{3} - y = \sqrt{3} - 5$; (*d*) $3x - y = 12$;
(*e*) $2x + y + 6 = 0$; (*f*) $x - y + 3 = 0$;
(*g*) $x + y + 4 = 0$; (*h*) $x - y\sqrt{3} = 5\sqrt{3}$;
(*i*) $x\sqrt{3} + y = 6$.

7. (*a*) $3^\circ 11'$; (*b*) $25^\circ 21'$; (*c*) $22^\circ 37'$; (*d*) $33^\circ 41'$;
(*e*) $19^\circ 26'$; (*f*) $73^\circ 1'$; (*g*) $124^\circ 31'$; (*h*) $127^\circ 53'$.

9. $x + y = \pm 8\sqrt{2}$. **11.** $x + y = 4$, or $9x + y = 12$.

13. 65. **17.** $x = \pm \frac{k}{4c}$, if the points are $(\pm c, 0)$.

19. (*a*) $x = \pm 3c$, if the vertices of the base are $(\pm c, 0)$;
(*b*) $y = \pm 3c$, if the vertices of the base are $(0, \pm c)$.

Pages 61, 62, 63

5. (*a*) $5x - 12y + 60 = 0$, $12x + 5y = 25$;
(*b*) $x + 2y = 20$, $2x - y = 10$;
(*c*) $x - 12y + 88 = 0$, $12x + y + 41 = 0$;
(*d*) $3x - 4y = 38$, $4x + 3y = 9$;
(*e*) $4x + 3y = 55$, $3x - 4y = 10$;
(*f*) $x - y + 2 = 0$, $x + y = 10$;
(*g*) $8x + 15y = 0$, $15x - 8y = 0$;
(*h*) $y = 0$, $x + 3 = 0$.

7. (*a*) $\frac{48}{\sqrt{61}}$; (*b*) $\frac{54}{\sqrt{113}}$; (*c*) $\frac{12}{5}$; (*d*) $\frac{16}{13}$;
(*e*) $\frac{24}{\sqrt{13}}$; (*f*) $\frac{12}{\sqrt{50}}$; (*g*) $\frac{18}{5}$; (*h*) 6.

9. (*a*) $y = 0$, $3x - y = 18$, $3x - 4y = 0$;
(*b*) $x = 3$, $x + 3y = 16$, $4x + 3y = 25$;
(*c*) $x = 8$, $x + 3y = 0$, $4x + 3y = 24$;
(*d*) $3x - 7y = 0$, $3x + 2y = 18$, $6x - 5y = 18$.

11. (*a*) 76; (*b*) 84.

13. The point of intersection is
(*a*) $(6, \frac{17}{5})$; (*b*) $(\frac{20}{3}, \frac{10}{3})$; (*c*) $(8, \frac{16}{5})$.

19. (*a*) $c = (16, -3)$, $r = \sqrt{265} = 16.28$;
(*b*) $c = (0, 10\frac{1}{2})$, $r = \frac{1}{2}\sqrt{697} = 13.20$;
(*c*) $c = (-4, -6)$, $r = \sqrt{208} = 14.42$;
(*d*) $c = (-\frac{10}{3}, -\frac{16}{3})$, $r = \frac{1}{3}\sqrt{884} = 9.91$.

21. $(10, 4)$, $(-6, -8)$, $x + 7y = 38$, $x + 7y = -62$, $7x - y = 66$, $7x - y = -34$.

23. (*a*) $y = -3x$ or $y = \frac{1}{3}x$; (*e*) $y = \frac{2}{3}x$ or $y = -\frac{3}{2}x$;
(*b*) $y = -\frac{1}{2}x$ or $y = 2x$; (*f*) $y = -\frac{1}{2}x$ or $y = 2x$;
(*c*) $y = -3x$ or $y = \frac{1}{3}x$; (*g*) $y = -\frac{3}{2}x$ or $y = \frac{2}{3}x$;
(*d*) $y = \frac{5}{7}x$ or $y = -\frac{7}{5}x$; (*h*) $y = \frac{1}{3}x$ or $y = -3x$.

Pages 67, 68

1. (*a*) $x + y = 6\sqrt{2}$; (*e*) $x - y = 16\sqrt{2}$;
(*b*) $x\sqrt{3} - y + 10 = 0$; (*f*) $x\sqrt{3} - y = 40$;
(*c*) $x + y\sqrt{3} = 24$; (*g*) $x + y\sqrt{3} + 20 = 0$;
(*d*) $y = 8$; (*h*) $x + 12 = 0$.

3. (*a*) $-\frac{4}{5}x + \frac{3}{5}y = -\frac{12}{5}$; (*e*) $-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 0$;
(*b*) $\frac{8}{17}x + \frac{15}{17}y = -\frac{12}{17}$; (*f*) $-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = -1$;
(*c*) $\frac{3}{5}x + \frac{4}{5}y = \frac{12}{5}$; (*g*) $\frac{x}{2} + \frac{\sqrt{3}}{2}y = -5$;
(*d*) $\frac{12}{13}x + \frac{5}{13}y = 0$; (*h*) $-\frac{3}{5}x + \frac{4}{5}y = -\frac{5}{2}$.

5. (*a*) 53° 8′; (*b*) 67° 23′; (*c*) 118° 4′; (*d*) 26° 34′;
(*e*) 63° 26′; (*f*) 157° 23′; (*g*) 104° 2′; (*h*) 45°.

7. (*a*) $x - y = 15$; (*c*) $24x + 7y = 50$;
(*b*) $x - \sqrt{3}y = 5 + 10\sqrt{3}$; (*d*) $3x + 4y + 25 = 0$.

9. $p = \frac{ab}{\sqrt{a^2 + b^2}}$; $\tan \omega = \frac{a}{b}$.

11. $\left(\frac{ab^2}{a^2 + b^2}, \frac{a^2b}{a^2 + b^2}\right)$.

Pages 70, 71, 72, 73

1. (*a*) 1.6; (*d*) 0;
(*b*) $2\sqrt{2}$ or 2.83^-; (*e*) $-\frac{15}{13}$ or -1.15^+;
(*c*) $\frac{19}{10}\sqrt{5}$ or 4.25^-; (*f*) $\frac{9}{2}\sqrt{2}$ or 6.36^+.

3. (*a*) $\sqrt{10}$ or 3.16^+; (*c*) $\frac{12}{13}\sqrt{13}$ or 3.33^-;
(*b*) $\frac{4}{13}\sqrt{13}$ or 1.11^-; (*d*) 4.8.

5. (*a*) $5\frac{3}{5}$; (*b*) $27\frac{3}{4}$; (*c*) $28\frac{1}{2}$.

7. (*a*) $94\frac{1}{2}$; (*b*) $17\frac{1}{2}$; (*c*) $52\frac{1}{2}$; (*d*) 56.

9. (*a*) $7x + 7y = 11$, $x - y = 1$;
(*b*) $7x - 7y = 15$, $x + y = 1$.

11. (*a*) $7x + 7y = 71$; (*b*) $7x - 7y + 45 = 0$.

13. $3x + 4y = 41$, $3x + 4y + 23 = 0$.

15. (*a*) $x - 3y = 0$, $2x + 4y = 25$, $7x - y = 50$;
(*c*) $(7\frac{1}{2}, 2\frac{1}{2})$, $2\frac{1}{2}$.

Pages **75**, **76**

3. (*a*) $b = \frac{5}{3}\sqrt{3}$, $m = \frac{1}{3}\sqrt{3}$, $p = 2\frac{1}{2}$, $\omega = 120°$;
(*c*) $a = 4\sqrt{3}$, $b = 12$, $m = -\sqrt{3}$, $\omega = 30°$;
(*d*) $b = -10\sqrt{3}$, $\alpha = 60°$, $m = \sqrt{3}$, $p = -5\sqrt{3}$;
(*h*) $a = \pm 13\frac{1}{3}$, $\alpha = 36° 52'$ or $143° 8'$, $m = \pm \frac{3}{4}$, $\omega = 126° 52'$ or $53° 8'$;
(*i*) $a = 6\sqrt{5}$, $b = 12\sqrt{5}$, $\alpha = 116° 34'$, $\omega = 26° 34'$.

5. (*a*) $(y + 1) = m(x - 5)$;
(*h*) $x \cos \omega + y \sin \omega = 13$;
(*j*) $x + y = k$.

7. (*a*) -2; (*d*) 9; (*f*) -25.

9. $12x - 5y = k$.

11. $y - 5 = m(x - 8)$, $-\frac{5}{8}$.

13. (*a*) Impossible; (*d*) Impossible; (*g*) $\pm \frac{1}{2}\sqrt{3}$;
(*b*) 48; (*e*) Impossible; (*h*) $-6\frac{1}{2}$.
(*c*) $-\frac{5}{36}$; (*f*) $-\frac{1}{2}$;

Pages **78**, **79**

1. $141x + 218y - 167 = 0$. **3.** $8y + 15 = 0$, $12x - 49 = 0$.

5. (*a*) $37x + 37y = 85$, $37x - 37y + 25 = 0$;
(*b*) $2x + 3y = 0$, $3x - 2y = 0$;
(*c*) $x = \frac{1}{11}$, $y = \frac{10}{11}$;
(*d*) $50x - 90y + 283 = 0$, $90x + 50y = 169$.

Page **81**

3. (*a*) $2\frac{1}{4}$; (*b*) $4\frac{1}{2}$; (*c*) 36.

7. (*a*) $x^2 - y^2 - 8x + 16 = 0$;
(*b*) $4x^2 - y^2 - 20x - 2y + 24 = 0$;
(*c*) $x^2 - 4xy + 4y^2 - 4x + 8y - 21 = 0$;
(*d*) $9x^2 - 16y^2 - 96y - 144 = 0$.

9. $4y - 3 = 0$, $2x + 3 = 0$.

11. $3x^2y - 5xy^2 - 15xy = 0$.

Pages **82**, **83**, **84**

1. (*a*) $x^2 + y^2 - 6x - 10y + 9 = 0$;
(*c*) $x^2 + y^2 - 8x + 6y = 0$.

3. (a) $x^2 + y^2 - 8x \pm 12y + 16 = 0$;
(b) $x^2 + y^2 \pm 16x + 12y + 36 = 0$;
(c) $x^2 + y^2 + 32x + 32y + 256 = 0$;
(d) $x^2 + y^2 + 16x \pm 12y = 0$;
(e) $x^2 + y^2 + 24x \pm 10y = 0$;
(f) $x^2 + y^2 + 4x + 2y = 20$, or
$x^2 + y^2 - 12x + 2y + 12 = 0$.

5. $x^2 + y^2 + 2x - 8y = 36$.

7. $25x^2 + 25y^2 - 300x + 20y = 696$.

Pages 86, 87, 88, 89

1. (a) $(-3, 4), 8$;
(b) $(-4, 5), 0$;
(c) $(2, -3), 6$;
(d) $(0, -8), 8$;
(e) $(\frac{1}{6}, 0), \frac{1}{6}$;
(f) $(-1, -4), 4$;
(g) $(4, 5), 6.40$;
(h) $(-\frac{5}{2}, \frac{3}{2}), 3$;
(i) $(12, -5), 13$;
(j) $(-1, -1), 0$.

5. (a) $(x+3)^2 + (y+4)^2 = 64$;
(b) $(x+4)^2 + (y+5)^2 = 0$;
(c) $x^2 + y^2 - 4x - 6y - 23 = 0$;
(d) $x^2 + y^2 - 16y = 0$; etc.

7. (a) $|k| \geq 4\sqrt{2}$; (b) $|k| \geq 2\sqrt{6}$.

9. (a) $(0, 2), (6, -4), x + y = 2$;
(b) $(6, 0), (-2, 8), x + y = 6$.

11. (a) $5x + 12y = 73$;
(b) $4x - 3y = 66$;
(c) $8x + 15y = 257$;
(d) $x - 3y + 26 = 0$.

13. $7x + y = 50$, 14.14.

15. $25x^2 + 25y^2 = 49$.

17. (a) $x^2 + y^2 = a^2$;
(b) $x^2 + y^2 + 2ay - a^2 = 0$, or $x^2 + y^2 - 2ay - a^2 = 0$;
(c) $kx^2 + ky^2 + 2ay = ka^2$, or $kx^2 + ky^2 - 2ay = ka^2$;
(d) $(x + 3a)^2 + y^2 = 4k^2$, or $(x - 3a)^2 + y^2 = 4k^2$.

19. $(k^2 - 1)(x^2 + y^2) + 2ax = a^2$.

21. $x^2 + y^2 + rx = 0$.

Pages 92, 93

1. (a) $x^2 + y^2 - 3x - 13y + 18 = 0$;
(b) $x^2 + y^2 - 6x + 8y = 0$;
(c) $8x^2 + 8y^2 - 79x - 32y + 95 = 0$;

(*d*) $4x^2 + 4y^2 - 8x + 19y - 140 = 0$;
(*e*) $x^2 + y^2 - 2x + 6y = 0$;
(*f*) $3x^2 + 3y^2 - 4x + 4y - 84 = 0$.

3. (*a*) $(x - 4)^2 + (y \pm 2\sqrt{3})^2 = 16$;
(*b*) $5x^2 + 5y^2 - 8x = 41$;
(*c*) $x^2 + y^2 - 6x + 4y + 8 = 0$;
(*d*) $x^2 + y^2 - 10x - 10y + 25 = 0$, or $x^2 + y^2 - 58x - 58y + 841 = 0$;
(*e*) $2x^2 + 2y^2 + 7x - 24y = 0$;
(*f*) $(x - 12)^2 + (y - 18)^2 = 160$, or $(x - 4)^2 + (y + 6)^2 = 160$;
(*g*) $x^2 + y^2 + 10x + 6y = 135$;
(*h*) $x^2 + y^2 + 8x - 12y = 117$;
(*i*) $(x - \frac{68}{5})^2 + (y + \frac{51}{5})^2 = 144$, or $(x + \frac{28}{5})^2 + (y - \frac{21}{5})^2 = 144$;
(*j*) $x^2 + y^2 - 10x - 16y = 11$, or $x^2 + y^2 - 6x + 12y = 55$;
(*k*) $x^2 + y^2 - 2x - 4y = 0$, or $9x^2 + 9y^2 - 58x + 4y = 0$;
(*l*) $x^2 + y^2 - 2x - 10y + 22 = 0$;
(*m*) $x^2 + y^2 + 4x - 6y = 0$, or $x^2 + y^2 + 44x - 14y + 208 = 0$.

Pages 99, 100

1. (*a*) 5; (*b*) 9.42; (*c*) 4.31; (*d*) 5.45; (*e*) 8.26; (*f*) 9.93; (*g*) 11.70; (*i*) 4.81; (*j*) 1.51.

3. (*a*) $(0, 7)$ and $(-1, 0)$; (*b*) $(1, 4)$ and $(4, -1)$.

5. (*a*) $(\frac{1}{4}, \frac{13}{8})$; (*b*) $(-3, -3)$.

9. $3x^2 + 3y^2 + 7x + 19y = 76$.

Pages 106, 107, 108, 109, 110

1. (*a*) $(3, 0)$, $x + 3 = 0$, 12;
(*b*) $(0, \frac{3}{2})$, $y + \frac{3}{2} = 0$, 6;
(*c*) $(0, -\frac{3}{2})$, $y = \frac{3}{2}$, 6;
(*d*) $(-\frac{2}{3}, 0)$, $x = \frac{2}{3}$, $2\frac{2}{3}$;
(*e*) $(-6, 0)$, $x = 6$, 24;
(*f*) $(\frac{1}{16}, 0)$, $x + \frac{1}{16} = 0$, $\frac{1}{4}$;
(*g*) $(0, 4)$, $y + 4 = 0$, 16;
(*h*) $(0, -3)$, $y = 3$, 12;
(*i*) $(0, a)$, $y + a = 0$, $4a$;
(*j*) $\left(\frac{1}{4k}, 0\right)$, $x + \frac{1}{4k} = 0$, $\frac{1}{k}$.

3. (*a*) $x^2 = 20y$; (*b*) $y^2 + 24x = 0$; (*c*) $y^2 = 4ax$;
(*d*) $3y^2 = 16x$, or $4x^2 + 9y = 0$;
(*e*) $y^2 = 16x$; (*f*) $y^2 + 12x = 0$.

5. $x^2 = 24y$.

7. $y^2 = 2ax$.

9. $2x^2 + 2y^2 - 5px = 0$.

11. $4p\sqrt{3}$.

13. $a + \frac{1}{2}p$.

15. $(0, 0)$, $(2p, 2p)$.

17. $4\,p$. **19.** $(-8,\ -8)$. **21.** $4\,x + 3\,y = 15$. **23.** $2\,y^2 = px$.

25. $x \pm 2\,y = 0$, or $x^2 - 4\,y^2 = 0$.

27. (*a*) $y^2 = 2\,px + p^2$, or $y^2 = -\,2\,px + p^2$;
(*b*) $x^2 = 2\,py - p^2$, or $x^2 = -\,2\,py - p^2$.

29. (*b*) $y = 2$, $(-5,\ 2)$, $(-7,\ 6)$ and $(-7,\ -2)$;
(*c*) $x = 3$, $(3,\ 2)$, $(11,\ 6)$ and $(-5,\ 6)$;
(*d*) $y = k$, $(h,\ k)$, $(h + a,\ k + 2\,a)$ and $(h + a,\ k - 2\,a)$.

31. (*b*) $y^2 + 8\,x - 4\,y + 44 = 0$;
(*c*) $x^2 - 6\,x - 16\,y + 41 = 0$;
(*d*) $(y - k)^2 = 4\,a(x - h)$.

33. (*a*) $y^2 - 20\,x + 4\,y + 24 = 0$, $y + 2 = 0$, 20;
(*b*) $x^2 + 2\,x + 14\,y + 64 = 0$, $x + 1 = 0$, 14;
(*c*) $y^2 + 8\,x - 6\,y + 25 = 0$, $y = 3$, 8;
(*d*) $x^2 - 2\,ax - 4\,by + a^2 + 4\,b^2 = 0$, $x = a$, $4\,b$;
(*e*) $y^2 - 2\,px - 2\,cy + c^2 + 2\,ap + p^2 = 0$, $y = c$, $2\,p$.

Pages 114, 115, 116

1. (*b*) $6,\ 3,\ 3,\ (\pm\,3\sqrt{3},\ 0),\ (\pm\,6,\ 0)$;
(*c*) $3,\ 2,\ 2\tfrac{2}{3},\ (\pm\,\sqrt{5},\ 0),\ (\pm\,3,\ 0)$;
(*d*) $\tfrac{4}{3},\ 1,\ \tfrac{3}{2},\ (\pm\,\tfrac{1}{3}\sqrt{7},\ 0),\ (\pm\,\tfrac{4}{3},\ 0)$;
(*e*) $\sqrt{30},\ \sqrt{5},\ \tfrac{1}{3}\sqrt{30},\ (\pm\,5,\ 0),\ (\pm\,\sqrt{30},\ 0)$;
(*f*) $5,\ \tfrac{5}{3},\ \tfrac{10}{9},\ (\pm\,\tfrac{10}{3}\sqrt{2},\ 0),\ (\pm\,5,\ 0)$;
(*g*) $2\tfrac{4}{5},\ 2,\ 2\tfrac{6}{7},\ (\pm\,\tfrac{4}{5}\sqrt{6},\ 0),\ (\pm\,2\tfrac{4}{5},\ 0)$;
(*h*) $8,\ 4,\ 4,\ (\pm\,4\sqrt{3},\ 0),\ (\pm\,8,\ 0)$;
(*i*) $5\sqrt{2},\ 5,\ 5\sqrt{2},\ (\pm\,5,\ 0),\ (\pm\,5\sqrt{2},\ 0)$.

3. (*a*) $9\,x^2 + 25\,y^2 = 900$;
(*b*) $x^2 + 3\,y^2 = 12$;
(*c*) $5\,x^2 + 9\,y^2 = 180$;
(*d*) $x^2 + 4\,y^2 = 64$;
(*e*) $25\,x^2 + 169\,y^2 = 4225$.

5. (*a*) $a = b\sqrt{2}$; (*b*) $a = 2\,b$.
7. $16\,x^2 + 25\,y^2 = 1600$.
9. $x^2 + 4\,y^2 = 36$.
11. $5\,x^2 + 9\,y^2 = 180$.
13. $b^2x^2 + a^2y^2 = a^2b^2$.

Pages 119, 120, 121

1. (*b*) $(0,\ \pm\,6),\ (0,\ \pm\,4\sqrt{2}),\ \tfrac{2}{3}\sqrt{2},\ \tfrac{4}{3}$;
(*c*) $(0,\ \pm\,\tfrac{10}{3}),\ (0,\ \pm\,\tfrac{5}{6}\sqrt{7}),\ \tfrac{1}{4}\sqrt{7},\ 3\tfrac{3}{4}$;
(*d*) $(0,\ \pm\,3\tfrac{1}{2}),\ (0,\ \pm\,\tfrac{7}{6}\sqrt{5}),\ \tfrac{1}{3}\sqrt{5},\ 3\tfrac{1}{9}$;
(*e*) $(0,\ \pm\sqrt{17}),\ (0,\ \pm\,3),\ \tfrac{3}{17}\sqrt{17},\ \tfrac{16}{17}\sqrt{17}$;

(*f*) $(\pm 2, 0)$, $(\pm \frac{8}{5}, 0)$, $\frac{4}{5}$, $\frac{36}{25}$;
(*g*) $(0, \pm 2)$, $(0, \pm \sqrt{3})$, $\frac{1}{2}\sqrt{3}$, 1.

3. (*a*) $16\,x^2 + 7\,y^2 = 112$;
(*b*) $169\,x^2 + 25\,y^2 = 4225$;
(*c*) $3\,x^2 + y^2 = 24$;
(*d*) $25\,x^2 + 16\,y^2 = 400$;
(*e*) $169\,x^2 + 25\,y^2 = 4225$.

5. $b = a\sqrt{1 - e^2}$. 7. $9\,x^2 + 5\,y^2 = 720$.

9. $4\,a^2x^2 + b^2y^2 = 4\,a^2b^2$.

11. $x^2 + n^2y^2 = a^2$, or $(n-1)^2x^2 + n^2y^2 = a^2(n-1)^2$.

13. $a^2x^2 + b^2y^2 = a^2b^2$.

15. (*a*) $3\,x^2 + 7\,y^2 = 115$;
(*b*) $9\,x^2 + 25\,y^2 = 225$, or $25\,x^2 + 9\,y^2 = 225$.

17. $a^2x^2 + b^2y^2 = a^2b^2$.

21. (*a*) $(4, 2)$, $(8, 2)$ and $(0, 2)$; $(4, 2 \pm 2\sqrt{3})$; 6; $\frac{1}{2}$;
(*b*) $(3, 1)$, $(3, 6)$ and $(3, -4)$; $(6, 1)$ and $(0, 1)$; $3\frac{3}{5}$; $\frac{4}{5}$.

Pages **125, 126**

1. (*b*) $2\sqrt{5}$, 4, $\frac{16}{5}\sqrt{5}$, $(\pm 6, 0)$;
(*c*) 3, 2, $\frac{8}{3}$, $(0, \pm \sqrt{13})$;
(*d*) $\frac{12}{5}$, 3, $7\frac{1}{2}$, $(\pm \frac{3}{5}\sqrt{41}, 0)$;
(*e*) 2, 6, 36, $(0, \pm 2\sqrt{10})$;
(*f*) 5, 2, $\frac{8}{5}$, $(\pm \sqrt{29}, 0)$;
(*g*) $\sqrt{10}$, $\sqrt{6}$, $\frac{6}{5}\sqrt{10}$, $(0, \pm 4)$;
(*h*) $\sqrt{15}$, $\sqrt{21}$, $\frac{14}{5}\sqrt{15}$, $(\pm 6, 0)$.

3. (*a*) $25\,x^2 - 64\,y^2 = 400$, or $25\,y^2 - 64\,x^2 = 400$;
(*b*) $5\,x^2 - 4\,y^2 = 80$, or $5\,y^2 - 4\,x^2 = 80$;
(*c*) $25\,x^2 - 24\,y^2 = 600$, or $25\,y^2 - 24\,x^2 = 600$;
(*d*) $16\,x^2 - 25\,y^2 = 400$, or $16\,y^2 - 25\,x^2 = 400$;
(*e*) $3\,x^2 - 4\,y^2 = 108$, or $3\,y^2 - 4\,x^2 = 108$;
(*f*) $16\,x^2 - 9\,y^2 = 324$, or $16\,y^2 - 9\,x^2 = 324$.

5. $3\,x^2 - y^2 = 3\,a^2$.

7. (*a*) $x^2 - y^2 + 9 = 0$; (*b*) $25\,x^2 - 7\,y^2 + 75 = 0$.

9. $(-5, -\frac{20}{3})$.

11. $2\,x^2 - 2\,y^2 + c^2 = 0$.

Pages 132, 133, 134

1. (b) $4, 6, \frac{1}{2}\sqrt{13}, (\pm 2\sqrt{13}, 0), 3x \pm 2y = 0;$
   (c) $8, 4, \frac{1}{2}\sqrt{5}, (0, \pm 4\sqrt{5}), 2x \pm y = 0;$
   (d) $5, 3, \frac{1}{5}\sqrt{34}, (\pm\sqrt{34}, 0), 3x \pm 5y = 0;$
   (e) $5, 5, \sqrt{2}, (0, \pm 5\sqrt{2}), x \pm y = 0;$
   (f) $3, 4, \frac{5}{3}, (0, \pm 5), 3x \pm 4y = 0;$
   (g) $5, 10, \sqrt{5}, (0, \pm 5\sqrt{5}), x \pm 2y = 0;$
   (h) $2\sqrt{6}, 2\sqrt{10}, \frac{2}{3}\sqrt{6}, (0, \pm 8), x\sqrt{3} \pm y\sqrt{5} = 0;$
   (i) $\sqrt{15}, 6, \frac{1}{5}\sqrt{85}, (\pm\sqrt{51}, 0), 2x\sqrt{3} \pm y\sqrt{5} = 0.$
3. (a) $25x^2 - 9y^2 = 225, \frac{1}{3}\sqrt{34}, (\pm\sqrt{34}, 0);$
   (b) $9x^2 - 4y^2 + 144 = 0, \frac{1}{3}\sqrt{13}, (0, \pm 2\sqrt{13});$
   (c) $4x^2 - y^2 = 64, \sqrt{5}, (\pm 4\sqrt{5}, 0);$
   (d) $9x^2 - 25y^2 + 225 = 0, \frac{1}{3}\sqrt{34}, (0, \pm\sqrt{34});$
   (e) $x^2 - y^2 = 25, \sqrt{2}, (\pm 5\sqrt{2}, 0);$
   (f) $9x^2 - 16y^2 = 144, \frac{5}{4}, (\pm 5, 0);$
   (g) $x^2 - 4y^2 = 100, \frac{1}{2}\sqrt{5}, (\pm 5\sqrt{5}, 0);$
   (h) $3x^2 - 5y^2 = 120, \frac{2}{5}\sqrt{10}, (\pm 8, 0);$
   (i) $12x^2 - 5y^2 + 180 = 0, \frac{1}{6}\sqrt{51}, (0, \pm\sqrt{51}).$
5. $2xy + 25 = 0.$ 7. $b.$ 9. $\dfrac{a^2b^2}{a^2 + b^2}.$
11. (a) $(3, 7)$ and $(3, -3)$, $(3, 6)$ and $(3, -2)$, $\frac{5}{4};$
    (b) $(10, -3)$ and $(-2, -3)$, $(4 \pm 2\sqrt{5}, -3)$, $\frac{3}{5}\sqrt{5}.$
13. (a) $3x^2 - y^2 = 27;$ (c) $x^2 - y^2 = 20;$
    (b) $4x^2 - 5y^2 + 80 = 0;$ (d) $4x^2 - y^2 + 20 = 0.$
15. $(c, \pm \frac{3}{2}c), (-c, \pm \frac{3}{2}c).$

Pages 137, 138, 139

1. (a) $4x'^2 - y'^2 + 4 = 0;$ (d) $x'^2 + y'^2 = 13;$
   (b) $2x'^2 + y' = 0;$ (e) $y' = x'^3 - 4x';$
   (c) $y'^2 = 8x';$ (f) $x'y' = c^2.$
3. (a) $x'^2 - y'^2 = 16;$
   (b) $x'y' + 8 = 0;$
   (c) $x'^2 - y'^2 = \pm a^2;$
   (d) $x'^2 - 3y'^2 = 9;$
   (e) $4x'^2 - 9y'^2 + 36 = 0;$
   (f) $(x'^2 + y'^2)(x' - y') = 2\sqrt{2}\,ax'y'.$

5. (*a*) $(-9, -1), (a-6, b-5)$;
(*b*) $(1, 2), (a+4, b-2)$;
(*c*) $(0, 8), (a+3, b+4)$;
(*d*) $(-3-h, 4+k), (a-h, b+k)$.

7. (*a*) $(-\sqrt{2}, 4\sqrt{2}), (5, -3)$;
(*b*) $(-2\sqrt{2}, 0), (2, 2)$;
(*c*) $[\frac{1}{2}\sqrt{2}(h-k), \frac{1}{2}\sqrt{2}(h+k)], (k, -h)$.

9. $x' = x \cos\theta + y \sin\theta,$
$y' = -x \sin\theta + y \cos\theta.$

Pages 141, 142, 143

1. (*a*) $(2, -2), 9x'^2 + 4y'^2 = 36$;
(*b*) $(2, 0), 16x'^2 - 25y'^2 = 400$;
(*c*) $(-4, 2), 4x'^2 - y'^2 = 0$;
(*d*) $(-\frac{1}{2}, 4), 4x'^2 + 5y'^2 = 80$;
(*e*) $(5, 12), x'^2 + y'^2 = 169$.

3. (*a*) $4x'^2 + y'^2 = 64$;
(*b*) $16x'^2 - y'^2 = 81$;
(*c*) $x'^2 = 8y'$;
(*d*) $y'^2 = 3x'$;
(*e*) $y' = x'^3 - \frac{7}{3}x'$.

5. (*a*) $y'^2 = 8x'$;
(*b*) $x'^2 = 12y'$;
(*c*) $y'^2 + 12x' = 0$;
(*d*) $x'^2 + 20y' = 0$.

7. (*a*) $9x'^2 + 25y'^2 = 225$;
(*b*) $16x'^2 + 7y'^2 = 112$.

9. (*a*) $4x'^2 + y'^2 = 36$;
(*b*) $4x'^2 + y'^2 = 36$.

11. (*a*) $y'^2 + 12x' = 0$;
(*b*) $3x'^2 + 4y'^2 = 48$;
(*c*) $3x'^2 - y'^2 = 48$;
(*d*) $x'^2 = 16y'$;
(*e*) $25x'^2 + 9y'^2 = 225$;
(*f*) $16x'^2 - 9y'^2 = 144$.

Pages 148, 149, 150

1. (*a*) $\frac{x^2}{25} + \frac{(y-3)^2}{16} = 1$; (*c*) $\frac{(x-2)^2}{36} + \frac{(y+2)^2}{20} = 1$;
(*b*) $\frac{(x-1)^2}{21} + \frac{(y-2)^2}{25} = 1$; (*d*) $\frac{(x+3)^2}{16} + \frac{(y-1)^2}{25} = 1$.

3. (*a*) $\frac{(x-2)^2}{25} - \frac{(y+1)^2}{20} = 1$;
(*b*) $\frac{(x-3)^2}{12} - \frac{(y-4)^2}{4} + 1 = 0$;
(*c*) $\frac{x^2}{16} - \frac{(y-2)^2}{20} = 1$;

(*d*) $\frac{(x+4)^2}{48} - \frac{(y-3)^2}{16} + 1 = 0.$

5. (*a*) $\frac{(x-2)^2}{25} - \frac{(y-3)^2}{9} + 1 = 0, \quad \frac{(x-2)^2}{9} - \frac{(y-3)^2}{25} = 1;$

(*b*) $\frac{(x+1)^2}{16} - \frac{(y-2)^2}{4} + 1 = 0, \quad \frac{(x+1)^2}{4} - \frac{(y-2)^2}{16} = 1;$

(*c*) $\frac{(x+4)^2}{4} - \frac{(y-6)^2}{16} + 1 = 0, \quad \frac{(x+4)^2}{16} - \frac{(y-6)^2}{4} = 1;$

(*d*) $\frac{(x+3)^2}{24} - \frac{(y+1)^2}{25} + 1 = 0, \quad \frac{(x+3)^2}{25} - \frac{(y+1)^2}{24} = 1.$

7. (*a*) $y^2 + 12(x-3) = 0;$
(*b*) $(y+4)^2 = 12(x-2);$
(*c*) $(x+5)^2 = 14(y+\frac{5}{2});$
(*d*) $(x-a)^2 + 8\,b(y-3\,b) = 0;$
(*e*) $(y-b)^2 + 4\,a(x-a) = 0.$

9. (*a*) $\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1;$

(*b*) $(x+1)^2 = 7(y-4);$

(*c*) $\frac{(x+3)^2}{24} - \frac{(y+2)^2}{16} = 1;$

(*d*) $(y+3)^2 = \frac{9}{2}(x-1);$

(*e*) $\frac{(x+2)^2}{32} + \frac{(y+\frac{1}{2})^2}{12} = 1;$

(*f*) $\frac{(x-\frac{1}{2})^2}{27} - \frac{(y+\frac{2}{3})^2}{12} + 1 = 0.$

13. (*a*) $y^2 = 8(x-2);$
(*b*) $x^2 + 4(y-4) = 0;$
(*c*) $x^2 = 5(y+5);$

(*d*) $\frac{x^2}{1} + \frac{(y+\frac{1}{2})^2}{\frac{1}{4}} = 1;$

(*e*) $\frac{x^2}{36} - \frac{(y-6)^2}{36} + 1 = 0;$

(*f*) $\frac{(x+2)^2}{4} - \frac{y^2}{1} = 1;$

(*g*) $\frac{(x+8)^2}{64} + \frac{y^2}{256} = 1;$

(*h*) $\frac{(x+8)^2}{64} - \frac{y^2}{256} = 1;$

(*i*) $\frac{x^2}{16} + \frac{(y-8)^2}{64} = 1.$

Pages 154, 155

1. $4\,x''^2 + y''^2 = 4.$
2. $3\,x''^2 + 2\,y''^2 = 6.$
3. $x''^2 - 4\,y''^2 + 4 = 0.$
4. $2\,x''^2 - 3\,y''^2 + 6 = 0.$
5. $2\,y''^2 = 3\,x''.$
6. $x''^2 + 4\,y'' = 0.$
7. $25\,x''^2 - y''^2 = 0.$
8. $4\,y''^2 = 1.$

9. $(x' + y')^2 + 1 = 0$. Imaginary.

10. $22\,x'^2 + 66\,x'y' + 110\,y'^2 + 199 = 0$. Imaginary.

11. $2\,x''^2 + y''^2 = 0$.

12. $9\,x''^2 + 4\,y''^2 = 0$.

13. $4\,x''^2 = 1$.

14. $x''^2 - y''^2 = 0$.

15. $9\,x''^2 + 4\,y''^2 = 36$.

16. $3\,x''^2 - y''^2 = 9$.

17. $x''^2 - y''^2 + 4 = 0$.

18. $x''^2 + 2\,y''^2 = 4$.

19. $y''^2 = 4\,x''$.

20. $x''^2 - y''^2 = \pm 2\sqrt{2}$.

21. (*a*) $9\,x''^2 - 16\,y''^2 + 28 = 0$;
(*b*) $3\,x''^2 - y''^2 = 8$.

Page 158

1. (*a*) $9\,x^2 + 16\,y^2 - 36\,x - 32\,y + 27 = 0$;
(*b*) $xy - 2\,x + y - 10 = 0$.

3. (*a*) $x^2 + 4\,xy + 4\,y^2 - 4\,x - 12\,y = 0$, or
$25\,x^2 + 60\,xy + 36\,y^2 - 100\,x - 108\,y = 0$;
(*b*) $x^2 - 2\,xy + y^2 - 2\,x - 1 = 0$.

5. (*a*) $4\,x^2 + xy + y^2 = 40$;
(*b*) $x^2 - 2\,xy + 2\,y^2 = 5$;
(*c*) $6\,x^2 - 13\,xy + 6\,y^2 = 0$.

7. (*a*) $2\,x = 16 - 19\,y + 5\,y^2$;
(*b*) $2\,x = 3\,y^2 - 5\,y$;
(*c*) $2\,x = 4 + y + y^2$;
(*d*) $x = y^2 - 1$.

Pages 163, 164

5. (*a*) $(8\sqrt{3}, 8)$;
(*b*) $(-2, 2\sqrt{3})$;
(*c*) $(0, 6)$;
(*d*) $(-4, 0)$;
(*e*) $(-\sqrt{2}, -\sqrt{2})$;
(*f*) $(4\sqrt{2}, -4\sqrt{2})$.

7. (*a*) $(2, 120°)$, $(-2, 300°)$;
(*b*) $(2\sqrt{2}, -45°)$, $(-2\sqrt{2}, 135°)$;
(*c*) $(13, -67° 23')$, $(-13, 112° 37')$;
(*d*) $(4, 60°)$, $(-4, 240°)$;
(*e*) $(-2, 45°)$, $(2, 225°)$;
(*f*) $(5, 53° 8')$, $(-5, 233° 8')$.

9. (*a*) $x^2 + y^2 = 16$;
(*c*) $x^2 + y^2 - 10\,x = 0$;
(*f*) $2\,x - 3\,y = 0$;
(*g*) $x^2 + y^2 - 2\,ay = 0$;
(*h*) $y = 2\,a$;
(*i*) $x^2 + y^2 - x - y = 0$.

10. (a) $\tan \theta = 2$;
(c) $\rho \cos \theta = a$;
(f) $\rho = 6 \cos \theta$;
(h) $\rho^2 \cos 2 \theta = 16$;
(k) $\rho^2 \sin 2 \theta = a^2$.

Pages 171, 172

1. (a) $\rho = - a \cot^2 \theta' \csc \theta'$;
(b) $\rho = - a \csc \theta' \pm b$;
(c) $\rho = a \cos 3 \theta'$;
(d) $\rho = - a \sin \theta'$;
(e) $\rho = a - b \sin \theta'$.

3. (a) $\rho = a(1 + \cos \theta)$;
(b) $\rho = a - b \cos \theta$;
(c) $\rho = - a \tan^2 \theta \sec \theta$;
(d) $\rho = 2 a \tan \theta \sec \theta$;
(e) $\rho = - a \cos 3 \theta$;
(f) $\rho = - a \cos \theta$.

Pages 174, 175

1. $\rho = \dfrac{\pm ep}{1 \mp e \cos \theta}$.

3. $2\,ep$.

5. (a) $\frac{64}{3}, \frac{32}{3}\sqrt{3}$;
(b) $18, 12\sqrt{2}$;
(g) $\frac{32}{5}, \frac{8}{5}\sqrt{15}$;
(i) $15, 10\sqrt{2}$;
(j) $\frac{64}{3}, \frac{32}{3}\sqrt{3}$.

9. (a) $\rho = \dfrac{ep}{1 + e \cos \theta}$;
(b) $\rho = \dfrac{ep}{1 + e \sin \theta}$.

Pages 177, 178

1. $\rho = 2(b + a \cos \theta)$, a limaçon.
2. $\rho = a(1 + \sin \theta)$, a cardioid.
3. $d = \sqrt{\rho_1^2 + \rho_2^2 - 2 \rho_1 \rho_2 \cos (\theta_1 - \theta_2)}$.
4. $\rho^2 + \rho_1^2 - 2 \rho_1 \rho \cos (\theta - \theta_1) = r^2$.
5. $\rho^2(1 - e^2 \cos^2 \theta) = b^2$.
6. $\rho = a(2 \cos \theta + 1)$, the trisectrix.
7. $\rho = a \sec \theta \pm b$, a conchoid.
8. $\rho \sin 2 \theta = a$.
9. $\rho = a \cos \theta$, where a is the radius of the given circle.
10. $\rho = a \sin 2 \theta$, the four-leafed rose.
11. $\rho = 2 a \tan \theta \sin \theta$, the cissoid.
12. $\rho^2 = 2 a^2 \cos 2 \theta$, the lemniscate.

Page 179

27. (a) $(x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)$;
(b) $(x^2 + y^2 - bx)^2 = a^2(x^2 + y^2)$;

(c) $x^3 = ay^2$;
(d) $(x - a)^2(x^2 + y^2) = b^2x^2$;
(e) $x^3 + xy^2 - 2\,ay^2 = 0$;
(f) $(x^2 + y^2)^3 = 4\,a^2x^2y^2$;
(g) $(x^2 + y^2)^2 = 2\,a^2xy$;
(h) $y = x \tan \dfrac{\sqrt{x^2 + y^2}}{a}$.

Pages 184, 185

5. (c) $x = -\log_3 y$;
(e) $x = -\frac{10}{3} \log_e y$;
(g) $x = 5 \log_e 10\,y$;
(h) impossible;
(i) $x = \sqrt{-\log_e y}$;
(j) impossible.

7. (c) $x = 4 + 10^y$; (e) $x = 4 - 10^y$; (g) $x = \pm\sqrt{e^y - 1}$.

9. (a) $x = \log_e (y \pm \sqrt{y^2 - 1})$;
(b) $x = \log_e (y + \sqrt{y^2 + 1})$;
(c) $x = a \log_e \left(\dfrac{y \pm \sqrt{y^2 - a^2}}{a}\right)$.

Pages 192, 193, 194

5. (a) $\sin(-x) = -\sin x$, $\sin(\pi \pm x) = \mp \sin x$,
$\sin(2\,\pi \pm x) = \pm \sin x$; etc.

7. (a) $2\,\pi$;
(c) $4\,\pi$;
(e) $4\,\pi$;
(g) no period;
(j) no period;
(n) no period.

Pages 199, 200

3. (a) $xy = 36$;
(b) $y^2 = 4\,x$;
(c) $2\,y = (x - 2)^3$;
(d) $y^2 = 4\,x - \dfrac{x^2}{2} + \dfrac{x^3}{64}$;
(e) $(y + 1)^2 = (x - 1)^3$;
(f) $b^2x^2 + a^2y^2 = a^2b^2$;
(g) $x^2 + 4\,y^2 = 400$;
(h) $x^2 + y^2 - 4\,x + 4\,y = 17$;
(i) $x^2 - y^2 = a^2$;
(j) $4\,x^2 - 25\,y^2 = 100$;
(k) $\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1$;
(l) $\dfrac{x^2}{a^2} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1$;
(m) $y = a\left(1 - \cos \dfrac{x}{a}\right)$;
(n) $4\,ax - y^2 = 4\,a^2$.

5. $x = 10 + \frac{1}{2}\,t\sqrt{3}$, $y = \frac{1}{2}\,t$.

7. (a) $x = 288\sqrt{2}\,t$, $y = 288\sqrt{2}\,t - 16\,t^2$, 10,368 ft., 2,592 ft.;
(b) $x = 200\sqrt{3}\,t$, $y = 200\,t - 16\,t^2$, 4,330 ft., 625 ft.;

(c) $x = 640\,t,\ y = 480\,t - 16\,t^2$, 19,200 ft., 3600 ft.;
(d) $x = 120\,t,\ y = 160\,t - 16\,t^2$, 1,200 ft., 400 ft.

9. 5796 feet measured along the slope.

11. $y = x \tan \alpha - \dfrac{16\,x^2}{v^2 \cos^2 \alpha}$.

13. $\rho = \dfrac{\theta}{\pi}$.

15. Every $\dfrac{\pi}{k}$ seconds.

Page 202

3. $x = r \text{ arc vers } \dfrac{y}{r} - \sqrt{2\,ry - y^2}$.

7. x – intercepts: $-0.78,\ +0.78,\ 62.06\ \ 63.61$, etc.
y – intercepts: $-2,\ 3.79$.

Page 205

1. $x = \frac{4}{3} R \cos \theta - \dfrac{R}{3} \cos 4\,\theta,\ y = \frac{4}{3} R \sin \theta - \dfrac{R}{3} \sin 4\,\theta$.

5. (a) $x = R(2 \cos \theta - \cos 2\,\theta),\ y = R(2 \sin \theta - \sin 2\,\theta)$;
(b) $x = \dfrac{R}{2}(3 \cos \theta - \cos 3\,\theta),\ y = \dfrac{R}{2}(3 \sin \theta - \sin 3\,\theta)$.

Page 209

1. (a) $(x^2 + y^2 + bx)^2 = a^2(x^2 + y^2)$;
(b) $(x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)$;
(c) $x^2 + y^2 = \dfrac{b^2x^2}{(x - a)^2}$;
(d) $y^2 = \dfrac{x^3}{2\,a - x}$;
(e) $(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$;
(f) $(x^2 + y^2)^2 = ay(3\,x^2 - y^2)$.

6. $\rho = -\dfrac{a \cos 2\,\theta}{\cos \theta}$.

7. $\rho^4 - 2\,a^2\rho^2 \cos 2\,\theta + a^4 - c^4 = 0$.

Pages 213, 214, 215

1. (a) $-\dfrac{y_1}{x_1}$;
(b) $3\,ax_1^2$;
(c) $\dfrac{x_1 - h}{2\,p}$;
(d) $\dfrac{b^2x_1}{a^2y_1}$;
(e) $-\dfrac{x_1 - h}{y_1 - k}$.

3. (*a*) $3, -\frac{1}{3}$;
(*b*) $\infty, 0$;
(*c*) $\frac{2}{3}\sqrt{3}, -\frac{1}{2}\sqrt{3}$;
(*d*) $-\frac{5}{12}\sqrt{5}, \frac{12}{25}\sqrt{5}$;
(*e*) $\frac{5}{12}, -\frac{12}{5}$;
(*f*) $-\frac{3}{4}, \frac{4}{3}$;
(*g*) $\frac{1}{4}, -4$;
(*h*) $\frac{2}{3}, -\frac{3}{2}$.

5. (*a*) $(1, 3), (-1, -3)$; (*b*) $(\sqrt{3}, \frac{1}{2}), (-\sqrt{3}, -\frac{1}{2})$;
(*c*) $(10, -5\sqrt{3}), (-10, 5\sqrt{3})$;
(*d*) impossible;
(*e*) $(\frac{13}{5}\sqrt{5}, -\frac{26}{5}\sqrt{5}), (-\frac{13}{5}\sqrt{5}, \frac{26}{5}\sqrt{5})$;
(*f*) $(\frac{5}{17}\sqrt{17}, -\frac{20}{17}\sqrt{17}), (-\frac{5}{17}\sqrt{17}, \frac{20}{17}\sqrt{17})$;
(*g*) $(10, -5)$; (*h*) $(27, 20\frac{1}{4})$.

7. (*a*) $8^\circ 8'$;
(*b*) $103^\circ 54'$;
(*c*) 90°;
(*d*) $153^\circ 26', 63^\circ 26'$.

9. $\left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2+b^2}}\right), \left(\frac{-a^2}{\sqrt{a^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2+b^2}}\right)$.

11. (*a*) arc tan $\frac{2e}{e^2-1}$; (*b*) arc tan $\frac{2e}{e^2-1}$.

13. $p\left(m+\frac{1}{m}\right)$.

Pages 218, 219

1. (*a*) $2x - 3y = 13, 3x + 2y = 0$;
(*b*) $2x - y = 9, x + 2y = 2$;
(*c*) $24x - y = 32, x + 24y = 386$;
(*d*) $3x + 2y = 24, 2x - 3y + 10 = 0$;
(*e*) $16x + 9y = 28, 9x - 16y = 100$;
(*f*) $x + y + 2 = 0, x - y + 6 = 0$;
(*g*) $x - 8y + 15 = 0, 8x + y = 10$;
(*h*) $9x - y + 28 = 0, x + 9y = 6$.

3. (*a*) $-\frac{9}{2}, -2$;
(*b*) $-\frac{1}{2}, -2$;
(*c*) $\frac{2}{3}, 384$;
(*d*) $-4, -9$;
(*e*) $\frac{9}{4}, \frac{64}{9}$;
(*f*) $-2, -2$;
(*g*) $16, \frac{1}{4}$;
(*h*) $\frac{1}{8}, 9$.

5. $4x - 3y = k$ where k has the value:
(*a*) $\pm 5\sqrt{13}$;
(*b*) $\pm 3\sqrt{41}$;
(*c*) $\pm \frac{8}{5}\sqrt{2}$;
(*d*) impossible;
(*e*) impossible;
(*f*) $\frac{32}{3}$;
(*g*) impossible;
(*h*) $\pm \frac{2}{3}\sqrt{903}$.

7. (*a*) $2x_1,\ p,\ \sqrt{4x_1^2+y_1^2},\ \sqrt{y_1^2+p^2}$;
   (*b*) $-\frac{a^2y_1^2}{b^2x_1},\ -\frac{b^2x_1}{a^2},\ \frac{y_1}{b^2x_1}\sqrt{a^4y_1^2+b^4x_1^2},\ \frac{1}{a^2}\sqrt{a^4y_1^2+b^4x_1^2}$;
   (*c*) $\frac{a^2y_1^2}{b^2x_1},\ \frac{b^2x_1}{a^2},\ \frac{y_1}{b^2x_1}\sqrt{a^4y_1^2+b^4x_1^2},\ \frac{1}{a^2}\sqrt{a^4y_1^2+b^4x_1^2}$;
   (*d*) $-\frac{y_1^2}{x_1},\ -x_1;\ \frac{y_1r}{x_1};\ r$.

9. (*a*) $y_1x+py=y_1(p+x_1)$;
   (*b*) $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$;
   (*c*) $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$;
   (*d*) $y_1x-x_1y=0$.

11. $x+y+\frac{p}{2}=0$, or $x-y+\frac{p}{2}=0$.

13. $\left(\frac{a^2}{\sqrt{a^2+b^2}},\ \frac{\pm b^2}{\sqrt{a^2+b^2}}\right),\ \left(\frac{-a^2}{\sqrt{a^2+b^2}},\ \frac{\pm b^2}{\sqrt{a^2+b^2}}\right)$.

15. a^2.

Pages 224, 225

1. (*a*) $x+8y=0$;
   (*b*) $x-18y=0$;
   (*c*) $x+2y=0$;
   (*d*) $8y=5$;
   (*e*) $9x+32y=0$;
   (*f*) $8x-9y=0$.

3. $mm'=-\frac{a^2}{b^2}$.

5. $y=\frac{p}{m}$.

7. $a=b\sqrt{2}$.

11. $4ab$.

Pages 239, 240, 241, 242

1. $A=\frac{x^2}{4}\sqrt{3}$.

3. $V=\frac{\pi x^2}{3}(3r-x)$.

5. $P=2x+\frac{200}{x}$.

7. $A=\frac{x}{2}(20-x)$.

9. $y=\frac{h}{r}(r-x)$.

11. $V=4x^2(21-x)$.

13. 0.366 cu. ft. per degree.

15. $L=30+0.0000326\,t$.

17. $T=212-0.002\,h$.

23. $A=1.72\,s^2$.

25. $C=\frac{1}{128}V^2$.

27. $pv=150$.

Pages 248, 249 *

1. $y = 0.41 + .00785\,x$.
2. $y = 109.95 + .0198\,x$.
3. $d = .00761 + .0000123\,t$.
4. $y = 70.31 + 0.300\,x$.
5. $T = 1.18 + 0.327\,L$.
6. $y = 33.65 - 0.51\,x$.
7. $y = 36.03 + 3.13\,x$.
8. $y = 100.00 + 0.108\,x$.
11. $L = 605 - 0.689\,T$.
12. $F = 4.22 + 0.0428\,w$.
13. $L = 999.63 + 0.0188\,T$.

Pages 253, 254 *

1. $y = 1.64 - 0.312\,x^2$.
2. $y = 0.0029 + .001646\,x^2$.
3. $y = 54.4 + 0.506\,x^2$.
4. $y = 9.9 + 0.371\,x^2$.
5. $y = 12.1 - \dfrac{15.8}{x}$.
6. $y = 6.08 + \dfrac{24.8}{x}$.
7. $y = 0.0496 - \dfrac{0.901}{x}$.
8. $y = 15.7 + \dfrac{37.5}{x}$.
9. $y = 1.818 - 0.00025\,x + 0.000917\,x^2$.
10. $y = 0.253 + 0.338\,x + 0.0137\,x^2$.
11. $y = 3.03 + 15.15\,x + 4.80\,x^2$.
12. $n = 0.34 + \dfrac{69.24}{i}$.
13. $W = 682 - 19.97\,h + 0.179\,h^2$.
14. $t = -\,0.14 + 2.02\,R + 0.206\,R^2$, or
    $t = 1.92\,R + 0.219\,R^2$.
15. $v = 4.265 + 0.739\,x - 0.1485\,x^2$.

Pages 257, 258 *

1. $y = 0.994\,x^{0.336}$.
2. $y = 0.326\,x^{0.600}$.
3. $y = 0.0181\,x^{-0.321}$.
4. $y = 9.97\,e^{0.0133x}$.
5. $y = 7.50\,e^{-0.000765x}$.
6. $y = 9.95\,e^{-0.00751x}$.
7. $N = 200\,e^{0.5t}$.
8. $D = 58.8\,h^{0.5}$.
9. $t = 0.0357\,d^{1.195}$.
10. $T = 63.0\,e^{-0.0719t}$.
11. $T = 0.001344\,R^{1.474}$.

* NOTE. In calculating these answers the values of the variables were substituted in the appropriate type of equation and the resulting equations were grouped exactly as in the model examples. A variation in this grouping will lead to somewhat different results. To determine which of several results is the best it is necessary to calculate the residuals.

Pages 264, 265

1. (*a*) $\rho = 5\sqrt{2}$, $\cos\alpha = \frac{3}{10}\sqrt{2}$, $\cos\beta = \frac{2}{5}\sqrt{2}$, $\cos\gamma = \frac{1}{2}\sqrt{2}$;
$\rho = 6$, $\cos\alpha = 1$, $\cos\beta = \cos\gamma = 0$;
$\rho = \sqrt{46}$, $\cos\alpha = -\frac{3}{23}\sqrt{46}$, $\cos\beta = \frac{1}{46}\sqrt{46}$,
$\cos\gamma = \frac{3}{46}\sqrt{46}$;
(*b*) $\rho = \sqrt{14}$, $\cos\alpha = -\frac{1}{7}\sqrt{14}$, $\cos\beta = \frac{3}{14}\sqrt{14}$,
$\cos\gamma = -\frac{1}{14}\sqrt{14}$; $\rho = 3\sqrt{10}$, $\cos\alpha = \frac{7}{30}\sqrt{10}$,
$\cos\beta = -\frac{2}{15}\sqrt{10}$, $\cos\gamma = \frac{1}{6}\sqrt{10}$; $\rho = \sqrt{2}$, $\cos\alpha = \frac{1}{2}\sqrt{2}$,
$\cos\beta = 0$, $\cos\gamma = \frac{1}{2}\sqrt{2}$;
(*c*) $\rho = 2$, $\cos\alpha = \cos\beta = 0$, $\cos\gamma = 1$;
$\rho = 5$, $\cos\alpha = 0$, $\cos\beta = \frac{3}{5}$, $\cos\gamma = -\frac{4}{5}$;
$\rho = 3$, $\cos\alpha = -\frac{2}{3}$, $\cos\beta = -\frac{2}{3}$, $\cos\gamma = \frac{1}{3}$.

3. (*a*) 94° 11′, 51° 11′, 34° 38′;
(*b*) 169° 15′, 6° 51′, 3° 53′;
(*c*) 90°, 24° 6′, 65° 54′.

7. (*a*) $24\frac{1}{2}$;
(*b*) $24\frac{1}{2}$.

9. $(4, \pm\frac{4}{3}\sqrt{23}, -\frac{8}{3})$.

13. $\sqrt{29}$, $5\sqrt{2}$, $\sqrt{59}$.

15. (*a*) $(-2, 3, 0)$;
(*b*) $(0, 0, -4)$.

Pages 270, 271

1. (*a*) $\frac{x}{\frac{5}{3}} + \frac{y}{-\frac{5}{4}} + \frac{z}{\frac{5}{12}} = 1$, $\frac{3x}{13} - \frac{4y}{13} + \frac{12z}{13} = \frac{5}{13}$, $(+-+)$;
(*b*) $\frac{x}{2} + \frac{y}{8} + \frac{z}{8} = 1$, $\frac{4x}{3\sqrt{2}} + \frac{y}{3\sqrt{2}} + \frac{z}{3\sqrt{2}} = \frac{8}{3\sqrt{2}}$, $(+++)$;
(*c*) $\frac{x}{-\frac{5}{2}} + \frac{y}{10} + \frac{z}{\frac{5}{4}} = 1$,
$-\frac{4x}{9} + \frac{y}{9} + \frac{8z}{9} = \frac{10}{9}$, $(-++)$;
(*d*) $\frac{x}{-\frac{4}{3}} + \frac{z}{\frac{5}{2}} = 1$, $-\frac{15x}{17} + \frac{8z}{17} = \frac{20}{17}$, $(-\pm+)$;
(*e*) $\frac{x}{9} + \frac{y}{2} + \frac{z}{-3} = 1$, $\frac{2x}{11} + \frac{9y}{11} - \frac{6z}{11} = \frac{18}{11}$, $(++-)$;
(*f*) $\frac{x}{-\frac{27}{4}} + \frac{y}{-\frac{27}{7}} + \frac{z}{\frac{27}{4}} = 1$, $-\frac{4x}{9} - \frac{7y}{9} + \frac{4z}{9} = 3$, $(--+)$.

3. (*a*) $4x + 3y - 24z = 12$;
(*b*) $3x - 4y \pm\sqrt{11}\,z = 30$;
(*c*) $14x - 9y + 8z = 13$;
(*d*) $3x - 4y + 8z + 30 = 0$.

5. (*a*) $60° 51'$; (*c*) $58° 46'$; (*e*) $75° 58'$;
(*b*) $51° 3'$; (*d*) $88° 53'$; (*f*) $111° 44'$.

7. (*a*) $36° 52'$, $53° 8'$, $90°$;
(*b*) $45°$, $90°$, $45°$;
(*c*) $54° 44'$, $54° 44'$, $54° 44'$;
(*d*) $48° 11'$, $131° 49'$, $109° 28'$;
(*e*) $143° 55'$, $72° 22'$, $59° 40'$;
(*f*) $149° 0'$, $73° 24'$, $115° 23'$.

9. (*a*) $-1\frac{3}{5}$; (*c*) $-\frac{11}{3}\sqrt{3}$; (*e*) $-47\frac{\sqrt{2}}{7}$;
(*b*) 0; (*d*) $1\frac{1}{3}$; (*f*) $-3\frac{5}{7}$.

11. (*a*) $(-2, 1\frac{1}{2}, -5)$; (*b*) $(1\frac{3}{7}, -4\frac{3}{7}, -5\frac{4}{7})$.

13. $5x - 5y + 2z - 9 = 0$, and $x + y - 1 = 0$.

14. 1.

Pages 274, 275

1. (*a*) $(-3\frac{2}{7}, -3\frac{6}{7}, 0)$, $(-\frac{7}{8}, 0, 3\frac{3}{8})$, $(0, 1\frac{2}{5}, 4\frac{3}{5})$;
(*b*) $(3\frac{5}{6}, \frac{2}{3}, 0)$, $(2\frac{1}{2}, 0, -2)$, $(0, -1\frac{1}{4}, -5\frac{3}{4})$;
(*c*) $(4, 3\frac{2}{3}, 0)$, $(-7, 0, -5\frac{1}{2})$, $(0, 2\frac{1}{3}, -2)$.

3. (*a*) $\frac{x + 3\frac{2}{7}}{5} = \frac{y + 3\frac{6}{7}}{8} = \frac{z}{7}$;
(*b*) $\frac{x - 2\frac{1}{2}}{2} = \frac{y}{1} = \frac{z + 2}{3}$.

5. (*a*) $x - 1 = -y = \frac{z + 5}{-2}$;
(*b*) $x = 1, y = -1$; (*c*) $x + y = 0, z = 2$;
(*d*) $\frac{x - 1}{2} = y + 1 = \frac{z - 2}{\pm\sqrt{11}}$;
(*e*) $2x - 3y + 2z = 9, 3x - y - z = 2$.

7. $48° 31'$.

9. (*a*) $\frac{14x + 19}{-32} = \frac{7y + 46}{-13} = z$;
(*b*) $\frac{x}{1} = \frac{15y - 4}{-3} = \frac{5z + 3}{1}$,
$\frac{x}{1} = \frac{3y - 28}{-11} = \frac{z - 13}{-5}$,
$\frac{x}{1} = \frac{3y + 40}{15} = \frac{z - 13}{-5}$,
$\frac{x}{1} = \frac{15y + 64}{23} = \frac{5z + 3}{1}$.

Pages 278, 279

1. (*a*) $(x-1)^2 = 16(y^2+z^2)$;
(*b*) $y^2 + z^2 = 2\,px$;
(*c*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$.

3. $z^2 = x^2 + y^2$.

5. $y^2 = 2\,px - p^2$.

7. (*a*) $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$;
(*b*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$.

9. $\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{a^2} + 1 = 0$.

Pages 286, 287

1. (*a*) $y^2 = 12\,z,\ x^2 = 12\,z,\ x^2 + y^2 = 0$;
(*b*) $y^2 + z^2 = 4,\ x^2 + 4\,z^2 = 16,\ x^2 + 4\,y^2 = 16$;
(*c*) $y^2 = 0,\ x^2 = 0,\ x^2 - 4\,y^2 = 0$;
(*d*) Imaginary, $x^2 - 4\,z^2 = 9,\ x^2 - 4\,y^2 = 9$;
(*e*) $y^2 + 9\,z^2 = 0,\ x^2 = 9\,z^2,\ x^2 = y^2$;
(*f*) $y^2 - z^2 = 4,\ 4\,x^2 - 9\,z^2 = 36,\ 4\,x^2 + 9\,y^2 = 36$.

3. (*a*) $y^2 = 12\,z - 16$;
(*b*) $y^2 + z^2 = 0$;
(*c*) $y^2 = 4$;
(*d*) $4\,y^2 + 4\,z^2 = 7$;
(*e*) $y^2 + 9\,z^2 = 16$;
(*f*) $y^2 - z^2 + \frac{28}{9} = 0$.